Problem 1 (Small Gain Theorem)
Consider a dynamical system \( T = \frac{3(s+\frac{1}{2})}{s^2+2s+4} \) in a positive feedback loop with \( \Delta = -\frac{1}{10.58} \approx -0.603 \). Does the loop satisfy the conditions of the small gain theorem? Hint: \( T(j\sqrt{2}) = 1.6583 \). Use Bode asymptotic magnitude ideas to provide a visual pictorial. Is the small gain condition necessary for stability? Is the closed loop system stable? Provide a \( \Delta \) with \( |\Delta(j\omega)| = \frac{1}{2} \) for all \( \omega \) which destabilizes the feedback loop. In general, when is the small gain condition a necessary condition?

Problem 2 (Stability Robustness Analysis)
Consider the LTI plant \( P_o = \frac{1}{s-1} \left[ \frac{100}{s+100} \right] \) and nominal LTI controller \( K = \frac{3(s+\frac{1}{2})}{s} \left[ \frac{100}{s+100} \right] \) within a classic negative feedback configuration. The nominal open loop transfer function is \( L_o = P_o K = \frac{3(s+\frac{1}{2})}{s(s-1)} \left[ \frac{100}{s+100} \right]^2 \). The associated nominal closed loop system is stable with dominant closed loop poles near \(-1 \pm j1\) and robustness margins:

- downward gain margin \( \downarrow GM \approx \frac{1}{2} \) measured at \( \omega_1 \approx \frac{1}{2} \);
- upward gain margin \( \uparrow GM \approx 200 \) measured at \( \omega_2 \approx 100 \), and
- phase margin \( PM \approx 59^\circ \) measured at \( \omega_y \approx 3 \).

We wish to examine the robustness properties of the above loop with respect to specific modeling error perturbations. Consider the actual plants:

1. \( P_1 = P_o \left[ \frac{p}{s+p} \right]^2 \) where \( p > 0 \) (nominally large) represents an uncertain pole location,
2. \( P_2 = P_o \left[ \frac{z}{s} \right] \) where \( z > 0 \) (nominally large) represents an uncertain RHP zero location, and
3. \( P_3 = P_o e^{-s\Delta} \) where \( \Delta > 0 \) (nominally small) represents an uncertain time delay.

Do the following for each of the above systems.

(a) Determine the parameter margin for the nominal design. For \( L_1 = P_1 K \), how small can \( p \) be made before the closed loop goes unstable? For \( L_2 = P_2 K \), how small can \( z \) be made before the closed loop goes unstable? For \( L_3 = P_3 K \), how large can \( \Delta \) be made before the closed loop goes unstable? For each case specify the critical value of the parameter as well as the imaginary closed loop poles that result when the critical parameter value is used.

(b) Now we wish to use small gain theorem ideas to estimate the the above parameter margins. We wish to do so by considering additive, multiplicative, feedback, and divisive modeling error characterizations. Comment on the conservatism of the answer obtained in (b) with respect to the exact answer obtained in (a). Support each answer with a relevant visualization of the test being performed. Specify which test gave the best (i.e. closest or least conservative) result.

Problem 3 (MBC Design)
Consider the LTI plant \( P = 1 \). Note: \( P \) has no dynamics; i.e. no \( A_p \) or \( B_p \), just \( C_p = 0 \) and \( D_p = 1 \). Show how to design a model based controller (MBC) \( K \) such that the closed loop system (1) is stable, (2) exhibits zero steady state error to sinusoidal reference commands \( A \sin(\omega_c t + \theta) \), (3) has closed loop poles at \(-\sigma_1 \pm j\omega_1 \) and \(-\sigma_2 \pm j\omega_2 \). Hint: Just set up the relevant equations. You need not solve them in class.

Problem 4 (Exploring the KF Problem)
(a) What can happen if stabilizability and detectability assumptions are violated?
(b) What are KF stability, stability robustness, and singular value properties?
(c) Can \( g(s) = \frac{\frac{1}{s+1} + \frac{1}{s+2}}{\frac{1000}{s+100}} \) be an KF loop? Explain.
(d) Can \( g(s) = \frac{\frac{1000}{s+1} + \frac{1000}{s+2}}{s+100} \) be an KF loop? Explain.
(e) Give a classical explanation why the following makes sense:

\[ \lim_{\mu \to 0^+} \Sigma_{\mu} = 0 \quad \text{if and only if} \quad G_{FOL} = [A, L, C] \text{ is minimum phase}. \]  
\[ \lim_{\mu \to 0^+} J_{min} = 0 \quad \text{if and only if} \quad G_{FOL} = [A, L, C] \text{ is minimum phase}. \]
(f) What can be said about $A$ if $\lim_{\mu \to \infty} H_\mu = 0$? Be careful! Is it true that $\lim_{\mu \to \infty} H_\mu = 0$ if and only if $A$ is stable ($\Re \lambda_i < 0$)? What can be said about $\lim_{\mu \to \infty} H_\mu$ for $g_{FOL} = \frac{1}{s^2}$?

**Problem 5 (LQR)**

In this problem, we examine the design of an LQR-servo for the non-minimum phase plant $P = \frac{z-s}{s-p} = [p, 1, z-p, -1]$ ($p, z > 0$). Use $p = 1$, $z = 20$. Two approaches are to be taken:

1. express plant state $x_p$, in terms of the plant output $y$;
2. augment plant with $\begin{bmatrix} 100 \\ s+100 \end{bmatrix}$ at output so that $y = x_2$ is a state variable.

Use $R = \rho > 0$. We wish to choose $\rho$ such that dominant closed loop poles are complex with a 5 sec settling time. (Comment on the feasibility of this.) Integral action is desired so that step reference commands are followed. Is integral augmentation necessary for this? Hint: For each case, sketch the structure of the final LQR-based feedback loop. Set up and solve the CARE for $K_\rho$. Compute the control gain matrix $G_\rho$.

For each approach, describe guaranteed properties at controls. Hint: Be careful with first approach. Are there guaranteed properties at error? Discuss. Given that input disturbances occur at the plant input, will this structure reject step input disturbances. Give a clear explanation.

**Extra Credit (LQR Loop Shaping)**

Clearly describe how the matrix $M$ can be used to shape the LQR loop $G_{LQ}$. Provide a block diagram as well as various selection methods.

**Problem 6 (KF)**

(a) Show how to design a Kalman filter for $g_{FOL} = \frac{10-s}{s-1}$. Hint: $A = [0 1; 0 1]$, $L = [0 1]^T$, $C = [z \ -1]$. Clearly state the problem definition and all assumptions.

(b) Use $\Theta = \mu > 0$ as your measurement noise intensity. Solve the FARE for the error covariance matrix $\Sigma_\mu$. Determine the associated filter gain matrix $H_\mu$.

(c) What do we get numerically for $\mu = 0.1$?

**Problem 7 (LQG/LTR)**

(a) What is the LQG problem statement?
(b) What is the solution?
(c) What are the resulting properties?
(d) Why is LTR needed? (e) State and prove LTRI.

**Problem 8 (Weighted $H_\infty$ Sensitivity Optimization)**

(a) What is the classic weighted $H_\infty$ sensitivity problem statement?
(b) Determine the transfer function matrix for the generalized plant $P$.
(c) Determine a state space representation for $P$.
(d) Describe how to select the weighting functions $W_i$.
(e) Does the problem always have a solution? Explain.

**Problem 9 (Stability Robustness Margin and Sensitivity Bounds)**

(a) Given upper bounds on the sensitivity and complementary sensitivity ($|S| < \alpha$, $|T| < \beta$), determine bounds on the upward gain margin $\uparrow GM$, downward gain margin $\downarrow GM$, and phase margin $PM$. Hint: Use Nyquist ideas.

(b) If $\alpha = 1$, what can be said about the upward gain margin and the open loop transfer function?
If $\beta = 1$, what can be said about the downward gain margin and the open loop transfer function?
In each case, what can be said about the phase margin?

(c) Suppose $L$ is the open loop transfer function.
If $L$ is non-minimum phase, what can be said about $||S||_{\mathcal{H}_\infty}$?
If $L$ is unstable, what can be said about $||T||_{\mathcal{H}_\infty}$?