Problem 1 (Small Gain Theorem)
Consider a dynamical system $T = \frac{100}{s+1}$ in a negative feedback loop with $\Delta = \frac{1}{s}$. Does the loop satisfy the conditions of the small gain theorem? Provide a visual pictorial. Is the small gain condition necessary for stability? Is the closed loop system stable? Provide a (stable) $\Delta$ with $|\Delta(j\omega)| = \frac{1}{s}$ for all $\omega$ which destabilizes the feedback loop. Repeat this using the smallest possible stable $\Delta$. In general, when is the small gain condition a necessary condition? State clearly when the small condition is necessary and sufficient for closed loop stability.

Problem 2 (Small Gain Theorem: Stability Robustness with Respect to Plant Uncertainty)
(I) Consider the plant $P = P_o \left[ \frac{z-1}{z} \right]$ where $P_o = \frac{1}{s(s+1)}$ and $z > 0$. Suppose that the controller $K = 1$ is in a negative feedback loop with $P$. (a) Determine the range for $z \in [z_1, z_2]$ such that the closed loop system is stable. Specify all relevant imaginary closed loop poles.
(b) What range for $z$ does the small gain theorem yield for a multiplicative uncertainty description? Specify all relevant frequencies. How does the small gain theorem result compare with the result obtained from the exact analysis?

(II) Consider the plant $P = P_o \left[ \frac{100^2}{s^2+25(s+100)^2} \right]$ where $P_o = \frac{10}{s+1}$ and $\zeta > 0$. Suppose that the controller $K = 1$ is in a negative feedback loop with $P$. (a) Using a multiplicative modeling error description, show how the small gain theorem can be used to determine the range for $\zeta \in [\zeta_1, \zeta_2]$ such that the closed loop system is stable. Specify all relevant frequencies.
(b) What range for $\zeta$ does the small gain theorem yield for a multiplicative uncertainty description? Specify all relevant frequencies. How does the small gain theorem result compare with the result obtained from the exact analysis?

(III) Consider a negative feedback loop with open loop system $L = PK = \frac{3(s+\frac{2}{5})}{s(s-p)}$ where $P = \frac{1}{s-\frac{2}{5}}$, $K = \frac{3(s+\frac{2}{5})}{s(s-p)}$, and $p > 0$. (a) Determine the range for $p \in [p_1, p_2]$ such that the closed loop system is stable. Specify all relevant imaginary closed loop poles.
(b) Show how the small gain theorem can be used to estimate this range. Use $P_o = \frac{1}{s}$ with a feedback uncertainty description. What does this yield? Specify all relevant frequencies. How does the small gain theorem result compare with the result obtained from the exact analysis?

(IV) Consider the plant $P = P_o \left[ \frac{a}{s+a} \right]$ where $P_o = \frac{2}{s+2}$ and $a > 0$. Suppose that the controller $K = 1$ is in a negative feedback loop with $P$. (a) Determine the range for $a \in [a_1, a_2]$ such that the closed loop system is stable. Specify all relevant imaginary closed loop poles.
(b) What range for $a$ does the small gain theorem yield for a multiplicative uncertainty description? How does the small gain theorem result compare with the result obtained from the exact analysis?

Problem 3 (Command Following, Disturbance and Noise Attenuation)
Consider the open loop transfer function $L = \frac{100}{s+1} \left[ \frac{s+10^4}{10^8} \right]^2 \left[ \frac{10^8}{s+10^8} \right]^2$.
(a) Discuss the stability of the closed loop system. Provide a supporting root locus.
(b) Determine all frequencies $\omega$ at which $|L| = 10$ (20 dB)?
(c) Determine all frequencies $\omega$ at which $|L| = 0.1$ (-20 dB)?
(e) For what frequencies will we have good sinusoidal steady state command following of sinusoidal reference commands?
(f) For what frequencies will we have good sinusoidal steady state output disturbance attenuation?
(g) For what frequencies will we have good sinusoidal steady state noise attenuation?

Problem 4 (Stability Robustness Margins and Sensitivity Bounds)
Suppose $L$ is the open loop transfer function, $S_\text{def} = \frac{1}{s+1}$, and $T_\text{def} = 1 - S$. Let $e = St$ and $y = Tr$.
(a) Given upper bounds on the sensitivity and complementary sensitivity ($|S| < \alpha$, $|T| < \beta$), determine bounds on the upward gain margin $\uparrow GM$, downward gain margin $\downarrow GM$, and phase margin $PM$. Hint: Use Nyquist ideas.
(b) If $\alpha = 1$, what can be said about the upward gain margin and the open loop transfer function? If $\beta = 1$, what can be said about the downward gain margin and the open loop transfer function? In each case, what
can be said about the phase margin?

(c) What can you conclude if \( \max_{x \in L^2} \| x \|_{L^2} = 2 \)? What if \( \max_{x \in L^2} \| x \|_{L^2} = 2 \)?

(d) Given \( \hat{G}_M, IGM, \) and \( PM \), determine lower bounds for \( \| S \|_{H^\infty} \) and \( \| T \|_{H^\infty} \).

(e) If \( \hat{G}_M = 10 \), what can be said about \( \| S \|_{H^\infty} \) and \( \| T \|_{H^\infty} \)? What if \( \hat{G}_M = 0.1 \)? What if \( PM = 45^\circ \)?

(f) Is it possible to have great values for \( \hat{G}_M, IGM, PM \), and still have a poor peak sensitivity? Explain.

(g) If \( L \) is non-minimum phase, what can be said about \( \| S \|_{H^\infty} \)? If \( L \) is unstable, what can be said about \( \| T \|_{H^\infty} \)? Explain.

Let \( z, p > 0 \) denote an open loop RHP zero and pole, respectively.

(h) What can be said about \( \min\{ \| S \|_{H^\infty}, \| T \|_{H^\infty} \} \)? What can be said if \( p \) and \( z \) are close to one another?

(i) Suppose that \( \| WS \|_{H^\infty} \leq 1 \) for \( W = \frac{z^2M_1}{M_2} \). Relate \( \omega_s \) to \( z \). What can be said when \( M_s = 1, 1.5, 2, \infty \)?

(j) Suppose that \( \| WT \|_{H^\infty} \leq 1 \) for \( W = \frac{s + \omega_l/M_2}{\omega_l} \). Relate \( \omega_l \) to \( z \). What can be said when \( M_t = 1, 1.5, 2, \infty \)?

**Problem 5 (Sinusoidal Steady State Analysis via Singular Values)**

Consider the two-input two-output system \( P = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \) where \( P_1 = \begin{bmatrix} 2s \n s+1 \end{bmatrix} \) and \( P_2 = \begin{bmatrix} 2s \n s+1 \end{bmatrix} \).

Let \( c_1 \) and \( c_2 \) denote the system outputs - error signals. Let \( d_1 \) and \( d_2 \) denote the system inputs - disturbances. (a) Plot the singular values for \( P \). (b) Provide an SVD sinusoidal analysis at \( \omega_0 = 1 \). Specify the maximally amplified and attenuated disturbances and the corresponding steady state errors. (c) Provide an SVD sinusoidal analysis at \( \omega_0 = 1 \). Specify the maximally amplified and attenuated disturbances and the corresponding steady state errors.

**Problem 6 (LQR Loop Properties)**

Consider an LQR loop \( G_{LQ} \) obtained via \((A, B, M)\) and \( R = \rho I \) with \( \rho > 0 \).

(a) Can \( G_{LQ}(s) = \frac{1}{s+1} \)? Explain.

(b) Can \( G_{LQ}(s) = \frac{s+1}{s} \)? Explain.

(c) Can \( G_{LQ}(s) = \frac{1}{s^2+2s+1} \)? Explain.

(d) Can \( G_{LQ}(s) = \frac{1}{s(s+1)} \)? Explain.

(e) Can \( G_{LQ}(s) = \frac{1}{(s+1)(s+10)(s+100)} \)? Explain.

(f) Can \( G_{LQ}(s) = \frac{1}{s^2+10} \)? Explain.

(g) Can \( G_{LQ}(s) = \frac{1}{s+1} \)? Explain.

(h) Can \( G_{LQ}(s) = \frac{100}{s(s+1)} \)? Explain.

(i) Let \( L \) denote an open loop transfer function. Suppose \( |L(j\omega)| = 1 \) and \( \angle L(j\omega) = -121^\circ \). Can \( G_{LQ}(s) = L \)? Explain.

(j) Suppose \( |L(j\omega)| = 1 \), \( \omega_2 = 1 \), and \( DM = 1 \) sec. Can \( G_{LQ}(s) = L(s) \)? Explain.

(k) Consider \( S = \frac{1}{(s^2+2s+4)(s+2)} \). Can \( S_{LQ}(s) = S \)? Explain.

(l) Consider \( T = \frac{1}{(s^2+2(0.1)s+1)(s^2+1)} \). Can \( T_{LQ}(s) = T \)? Explain.

(n) Suppose that \( A \) is unstable. Let \( G = G_\rho \). What can be said about \( \lim_{\rho \rightarrow \infty} G_\rho \)?

(o) Let \( K = K_\rho \). When will we have \( \lim_{\rho \rightarrow 0} K_\rho = 0 \)?

(p) Let \( J_{min} = J_{min_\rho} \). When will we have \( \lim_{\rho \rightarrow 0} J_{min_\rho} = 0 \)?

(q) Suppose that \( \lim_{\rho \rightarrow 0} G_\rho = 0 \). Does this imply that \( \lim_{\rho \rightarrow 0} G_\rho = 0 \)? Explain.

(r) Let \( G = G_\rho \). Give conditions under which \( \lim_{\rho \rightarrow 0} \sqrt{\rho} G_\rho = WM \) where \( W \) is some orthonormal matrix. When is this result used?

**Problem 7 (Inner-Outer Loop LQR Servo)**

Consider \( P = \frac{1}{s^2+2s+1} \) where \( p, z > 0 \). Let \( u \) denote its input, \( x_p \) its state, and \( y \) its output. (a) Show how to construct an LQR servo which guarantees zero steady state error to step reference commands. Sketch a supporting block diagram. Do NOT compute the control gain matrix \( G \). (b) Compute the open loop transfer function \( L_u \) at the controls. Determine the associated upward and downward gain margins. What
constraints must the gains satisfy in order to meet the LQR margins? (c) Compute the open loop transfer function \( L_e \) at the error. Determine the associated upward and downward gain margins.

**Problem 8 (Kalman Filter: State Estimator)**
(a) Show how to design a Kalman filter for \( P = [A, B, C] \). Assume \( \Xi = I \), and \( \Theta = \mu > 0 \).
(b) What are the associated estimator equations?
(b) Sketch a block diagram for the estimator.
(c) Determine the state space representation for the state estimation error \( \hat{x} \). Suppose \( A = a, B = b, L = l, C = c \) are all scalars.
(d) Determine the the transfer function matrices: \( T_{\xi \hat{z}}, T_{\theta \hat{z}}, T_{y \hat{y}}, T_{y y} \).
(e) Determine \( \Sigma, H, G_{KF}, CLP = A - HC, T_{\xi \hat{z}}, T_{\theta \hat{z}}, T_{y \hat{y}}, T_{y y} \).
(f) Approximate each of the quantities in (e) and (f) for small and large \( \mu \).

**Problem 9 (Control System Design)**
Consider a plant \( P = \frac{1}{s} \left[ \frac{100}{s^{0.1} + 100} \right] \). Design a control system such that the closed loop system (a) is stable, (b) exhibits zero steady state error to output disturbances, (c) exhibits a settling time of approximately 5 sec, (d) exhibits an overshoot of approximately 1.5%, (e) attenuates the impact of high frequency sensor noise on the controls. Support your design with a rough root locus plot. Show how to estimate the associated gain and phase margins.

**Problem 10 (LQG/LTR)**
(a) Describe the LQG design process. What are the associated LQG robustness properties? Why are loop transfer recovery ideas important? (b) Show how to achieve loop transfer recovery (LTR) at the plant input. (c) State and prove the loop transfer recovery result.