Problem 1 (Gaussian Elimination, Fundamental Subspaces, General Solution, Projections)
Consider the system of linear algebraic equations:

\[ Ax = b \]

with \( A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 3 & 3 & 6 & 3 & 3 \end{bmatrix} \) and \( b = \begin{bmatrix} 8 \\ 16 \\ 24 \end{bmatrix} \).

(a) Parameterize the set of all solutions \( x \).
(b) Determine a basis for each of the four (4) fundamental subspaces: \( \mathcal{R}(A), \mathcal{R}(A^T), \mathcal{N}(A), \mathcal{N}(A^T) \).
(c) Determine a solution that lies within the range space of \( A^T \). What can be said about such a solution?
(d) Given that \( b = [b_1 \ b_2 \ b_3]^T \) does not lie within the range space of \( A \), determine the set of all \( x \) which minimizes the Euclidean norm (distance) \( \| b - Ax \| \). Amongst these, how would you determine the minimum norm solution?

Problem 2 (State Space Realization and Arithmetic)
(a) Provide two distinct state space realizations for the following LTI system

\[ H(s) = \frac{30s^4 + 10}{6s^4 - 5}. \]  

Sketch a block diagram for each realization. Indicate all state variables on your diagrams.
(b) Determine a state space representation for the feedback system with external signals \((r, d_i, d_o, n)\), state \( x = [x_p \ x_k]^T \), and outputs \((e, u, y)\) defined by the following equations:

\[ \begin{align*}
e &= r - y - n \\ u_p &= u + d_i \\ \dot{x}_k &= A_k x_k + B_k e \\ \dot{x}_p &= A_p x_p + B_p u_p \\ u &= C_k x_k \\ y &= C_p x_p + d_o.
\end{align*} \]

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Repeat (b) when \( u = C_k x_k + D_k e \) and \( y = C_p x_p + D_p u_p + d_o \). Assume that \( M \overset{\text{def}}{=} (I + D_p D_k)^{-1} \) is well defined.

Problem 3 (Modal Analysis)
Consider the linear dynamical system:

\[ \dot{x} = Ax \quad x(0) = x_o \]

with \( A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -25 \\ 0 & 1 & 0 & -31 \\ 0 & 0 & 1 & -7 \end{bmatrix} \). (a) Determine the system’s characteristic equation? (b) Given that \( s = -3 + j4 \) is an eigenvalue, determine all other eigenvalues of \( A \). (c) Determine a set of linearly independent eigenvectors for \( A \). Is \( A \) diagonalizable? (d) Show how you would compute \( x \) for a given initial condition. (Just explain main ideas.) (e) Determine how each mode can be independently excited via carefully selected (real) initial conditions. Determine the response for each initial condition.

Problem 4 (State Computation)
Consider the linear system:

\[ \dot{x} = Ax + Bu \]

with \( A = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \). Sketch a block diagram for the system. Compute \( x \) when \( u = \delta(t) \) (unit impulse function) and \( x(0) = [0 \ 0 \ 0]^T \).
**Problem 5 (Controllability, Pole Placement)**

Consider the LTI system defined by the state space representation:

\[ \dot{x} = Ax + Bu \]

with \( A = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

(a) Sketch a block diagram for the system.
(b) Is the system controllable?
(c) Can full state feedback be used to place the closed loop poles at -1, -2, -3? Explain.
(d) Can full state feedback be used to place the closed loop poles at 0, -2, -3? Explain.

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(e) Compute the system's characteristic equation, eigenvalues, right eigenvectors, left eigenvectors.
(f) Does there exist a left eigenvector of \( A \) that lies in the left null space of \( B \)? If so, what does this imply?
(g) Determine the set of all states that are reachable from \( x(0) = 0 \).
(h) What is the range of the controllability matrix? Relate your answer to your answer in (g).
(i) Can \( x = [1 \ 0 \ 0]^T \) be reached from \( x(0) = 0 \)? If so, how? If not, determine what closest state \( x(1) \) can be reached at \( t = 1 \)? Hint: consider a projection (i.e., least squares).
(j) Given that \( x(0) = 0 \), show that \( x(1) = \int_0^1 e^{A(1-\tau)}Bu(\tau)d\tau \). Compute \( W = \int_0^1 e^{A(1-\tau)BB^HE^{A^H(1-\tau)}}d\tau \). Is \( W \) invertible? Determine a basis for the range of \( W \). Show how this can be used to achieve the desired result in (i)? Hint: form \( x(1) = Wz \) and take \( u(t) = BHe^{A^H(1-t)}z \).

**Problem 6 (SVD)**

Construct a matrix \( M \) which

(1) maps the unit vector \( v_1 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \) to the vector \( 10u_1 = 10 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \);

(2) maps the unit vector \( v_2 = \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix} \) to the vector \( 0.1u_2 = 0.1 \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix} \).

Show pictorially how the matrix \( M \) maps the unit circle onto an ellipse.

**Problem 7 (Model Based Compensator)**

Consider the linear time invariant (LTI) plant

\[ P(s) = \frac{10 - s}{s - 3} \]

with \( A_p = 3, B_p = 7, C_p = 1, D_p = -1 \) and variables \((u_p, x_p, y_p)\). Design a model based compensator which satisfies the following design specifications:

(i) zero steady state error to step reference commands,
(ii) closed loop poles at \( s = -1 \pm j1, s = -20 \pm j20 \).

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Determine the final compensator \( K \). Use MATLAB to verify that your design achieves the desired closed loop poles. Let \( L = PK \) denote the open loop transfer function. Use MATLAB to sketch Bode magnitude and phase plots for \( L \). Let \( S = \frac{1}{s+1}, T = 1 - S = \frac{s}{s+1} \). Use MATLAB to plot the magnitude responses \(|S|, |T|, |KS|, |SP|\). Use MATLAB to determine the upward gain margin \( \uparrow GM \), downward gain margin \( \downarrow GM \), phase margin \( PM \), and delay margin \( DM \) for your design? Plot the response (output \( y \) and control \( u \)) to a unit step reference command. Explain why your design achieves the steady state specification. Give the main reason for the overshoot in the output \( y \). How is the response enhanced if you use a command pre-filter \( W = \frac{1}{s+2} \) where \(-z\) is a zero of the compensator \( K \).

**HOME:**

Do Problems 5, 6, 7 from EEE582, Exam 1, Fall 2006.
Do Problems 4, 5, and 6 from EEE582, Exam 2, Fall 2006.