ZOH-Equivalent or Step-Invariant Discretization / Transformation of Continuous-Time $K(s)$

- The following terms are used interchangeably in the literature:

ZOH-Equivalent Discretization $K_d(z)$ of $K(s)$

" " Transformation " " "

Step-Invariant Discretization $K_d(z)$ of $K(s)$

" " Transformation " " "
\[ K_d(z) \triangleq S \cdot K(s) \cdot H = \mathcal{Z} \{ \text{sampler} \cdot \mathcal{Z}^{-1} \left\{ K(s) \cdot \left(1 - \frac{e^{-sT}}{s} \right) \right\} \} \]

\[ = (1 - z^{-1}) \mathcal{Z} \{ \text{sampler} \cdot \mathcal{Z}^{-1} \left\{ K(s) \cdot \frac{z-1}{s} \right\} \} \]

**ZOH-Equivalence**

- \[ K_d(z) \] and \[ S \cdot K(s) \cdot H \] have the same discrete-time response to \( \delta[n] \) ... (by definition)

**Step-Invariance**

- Sampled continuous-time step response of \( K(s) \) is identical to discrete-time step response of \( K_d(z) \)

\[ K_d(z) \triangleq S \cdot K(s) \cdot H = \mathcal{Z} \{ \text{sampler} \cdot \mathcal{Z}^{-1} \{ u_k(t) - u_k(t-T) \} \} \]

\[ = \mathcal{Z} \{ \text{sampler} \cdot \mathcal{Z}^{-1} \left\{ K(s) \cdot \left(1 - \frac{e^{-sT}}{s} \right) \right\} \} \]
Example

a) Determine the ZOH equivalent or step-invariant discretization for \( K(s) = \frac{1}{s} \)

Solution:

\[
\begin{align*}
1 \rightarrow K(s) \rightarrow u_k(t) = \frac{1}{T} \leftrightarrow U_k(s) &= \frac{1}{s^2} \\
&= \mathcal{Z}^{-1}\left\{ \frac{K(s)}{s} \right\} \\
\text{simplified} \quad \mathcal{Z}^{-1}\left\{ \frac{K(s)}{s} \right\} &= n T \{1[n]\} \\
\mathcal{Z}\{1[n]\}\text{ sampled} \quad \mathcal{Z}^{-1}\left\{ \frac{K(s)}{s} \right\} &= \frac{T}{(z-1)^2} \\
K_d(z) &= \left(1 - \frac{1}{z-1}\right) \sum \left\{ \text{simplified} \quad \mathcal{Z}^{-1}\left\{ \frac{K(s)}{s} \right\} \right\} = \frac{T}{z-1}.
\end{align*}
\]

Recall:

\[
\begin{align*}
a^n 1[n] &\leftrightarrow \frac{z}{z-a} \\
n^{-1} 1[n] &\leftrightarrow \frac{z}{(z-a)^2} \\
n 1[n] &\leftrightarrow \frac{z}{(z-1)^2}.
\end{align*}
\]

Same result obtained via Forward-Difference Backward Euler:

\[
\begin{align*}
s &= \frac{z-1}{T} \\
z &= 1 + sT.
\end{align*}
\]

b) Show that \( K(s) \approx K_d(e^{sT}) \) for small \( sT \).

Solution:

\[
K_d(e^{sT}) = \frac{T}{e^{sT} - 1} \approx \frac{T}{1 + sT - 1} = \frac{T}{s} = K(s) \quad \checkmark
\]

Problem

How small must \( T \) be so that

\[
| K(j\Omega) - K_d(e^{j\Omega T}) | < 0.1
\]

for all \( |\Omega| < 10 \) ? Hint: Use MATLAB.
Example

a) Determine ZOH-equivalent or step-invariant discretization for \( K(s) = \frac{b}{s+a} \)

**Solution:**

Step Response:

\[
U_k = \frac{b}{s(s+a)} = \frac{b}{s} + \frac{-b}{s+a}
\]

\( u_k(t) = \frac{b}{a} \left[ 1(t) - e^{-at} u(t) \right] \)

Sample:

\( u_k[n] = u_k[nT] = \frac{b}{a} \left[ u[n] - (e^{-aT})^n u[n] \right] \)

\( z \)-Transform:

\[
U_k[z] = \frac{b}{a} \frac{z}{z-1} - \frac{b}{a} \frac{z}{z-e^{-at}}
\]

\[
= \frac{b}{a} \left[ \frac{z-e^{-at}}{(z-1)(z-e^{-at})} \right]
\]

\[
= \frac{b}{a} (1-e^{-at}) \left[ \frac{z}{z-1} \right] \left[ \frac{1}{z-e^{-at}} \right]
\]

\( K_d(z) = \left( \frac{z-1}{z} \right) U_k[z] \text{ sampled } L\left\{ \frac{K(s)}{s} \right\} = \frac{b}{a} (1-e^{-at}) \frac{z}{z-e^{-at}} \)

b) Show that \( K(s) \approx K_d(e^{sT}) \) for small \( sT \)

**Solution:**

\[
K_d(e^{sT}) = \frac{b}{a} (1-e^{-at}) \frac{e^{sT}}{e^{sT}-e^{-at}} = \frac{b}{a} (1-e^{-(sT-1)}) \frac{e^{sT}}{e^{sT} - e^{-(sT-1)}} \approx \frac{b}{a} \left( \frac{e^{-(sT-1)}}{(sT)T} \right) = \frac{b}{a} \frac{e^{-aT}}{(sT)T} = \frac{b}{s+a} \checkmark
\]

**Problem:**

Let \( b=10 \) and \( a=1 \).

How small must \( T \) be so that

\[
|K(j\omega) - K_d(e^{j\omega T})| < 0.1
\]

for all \( |\omega| \leq 100 \) ? **Hint:** Use MATLAB.
Example

a) Determine the ordinary differential equation for \( K(s) = g \left( \frac{s+b}{s+a} \right) = \frac{U}{E} \).

b) Determine the DT-equivalent or step-invariant discretization for \( K(s) \).

c) Determine the ordinary difference equation for \( Kd(z) = \frac{Ud}{Ed} \).

d) Compute \( |Kd(e^{j\omega T})| \) and \( |K(j\omega)| \).

e) Show \( K(s) \approx Kd(e^{sT}) \) for small \( sT \).

f) Let \( g = 3 \), \( a = 0 \), \( b = \frac{2}{3} \), and \( T = \frac{2\pi}{36} \approx 0.2 \).

Plot \( |K(j\omega)| \) and \( |Kd(e^{j\omega T})| \)

Over what frequency range is \( |Kd(e^{j\omega T}) - Kd(e^{j\omega T})| \leq 0.1 \)?