Problem 1 (Analysis of a Continuous-Time Feedback System)

Consider the feedback system defined by the continuous-time LTI plant \( P = \left[ \frac{1}{s+1} \right] \left[ \frac{100-s}{s+100} \right]^2 \), series compensator \( K = \left[ \frac{\frac{1}{2}(s+\frac{1}{2})}{s} \right] \), pre-filter \( W = \left[ \frac{1}{s+\frac{1}{2}} \right] \), and sensor dynamics \( H = 1 \).

(a) Determine the (approximate) closed loop transfer functions \( T_{rP}, T_{dP} \). What are the closed loop poles, the associated time constant \( \tau \), settling time \( t_s \), undamped natural frequency \( \omega_n \), damping factor \( \zeta \), percent overshoot \( M_p \), time to peak \( t_p \)?
(b) What is the (approximate) linear ordinary differential equation relating \( y \) to \( r \) and \( d \)?
(c) Approximate the output \( y \) when \( r = 5 + \cos(0.01t - 45^\circ) \) and \( d_i = 2 - \sin(0.01t + 53^\circ) \). Show how to compute all coefficients.
(d) Approximate the steady state output \( y_{ss} \) for the scenario in (c).
In what follows, use all poles and zeros!
(e) Sketch a carefully labeled root locus. Determine all imaginary crossovers and asymptote angles.
(f) Sketch Bode asymptotic magnitude and phase plots for \( L \). Show gain and phase margins on your plots.
(g) Compute all margins; i.e. upward and downward gain margins, phase margin, delay margin.
(h) From the margins computed in (g), determine lower bounds on the peak sensitivity \( S \) and complementary sensitivity \( T \).

Problem 2 (Analysis of a Discrete-Time Feedback System)

Suppose that the above system is discretized with a sampling rate \( \omega_s = 600 \) rad/sec \( (T = 0.010472 \) sec) using Tustin’s discretization method (trapezoidal rule/bilinear transformation). Consider the feedback system defined by the discrete-time LSI plant \( P_d = 0.00051462 \left[ \frac{z+1}{z-0.9935} \right] \left[ \frac{z-3.19}{z-0.312} \right]^2 \), series compensator \( K_d = 2.5082 \left[ \frac{z-0.9935}{z-1} \right] \), pre-filter \( W_d = 0.0032622 \left[ \frac{z+1}{z-0.9935} \right] \), and sensor dynamics \( H_d = 1 \).

(a) Determine the (approximate, “low frequency”) closed loop transfer functions \( T_{rP}, T_{dP} \). What are the approximate closed loop poles? 1% settling time? Compare settling time with that in Problem # 1.
(b) What is the (approximate) linear ordinary differential equation relating \( y \) to \( r \) and \( d \)?
(c) Approximate the output \( y_n \) when \( r_n = 10 \) and \( d_{in} = \cos(0.01nT - 60^\circ) \). Show how to compute all coefficients.
(d) Approximate the steady state output \( y_{ss} \) for the scenario in (c).
In what follows, use all poles and zeros!
(e) Sketch a carefully labeled root locus for the above system. Determine all unit circle \( (|z| = 1) \) crossovers.
(f) Sketch \( |L_d| \) versus frequency. Compare with \( |L| \) from Problem # 1.

Problem 3 (Discretization)

Consider the continuous-time LTI open loop system \( L = \left[ \frac{2(s+1)}{s^2} \right] \left[ \frac{100}{s+100} \right]^2 \left[ \frac{10^3-s}{s+10^3} \right]^2 \).

(a) Sketch a root locus. Determine imaginary crossovers. (Hint: A Bode phase plot could help!) What are the (approximate) dominant closed loop poles? settling time?
In this problem, you will be discretizing \( L \) using a sampling frequency \( \omega_s = 600 \) rad/sec \( (T \approx 0.0105) \).
(b) Discretize \( L \) using the bilinear transformation.
(c) Sketch a discrete-time root locus for the \( L_d \) obtained in (b). Determine unit circle crossovers. What are the (approximate) dominant closed loop poles?
(d) Sketch \( |L(j\omega)| \) and \( |L_d(e^{j\omega T})| \) versus frequency \( \omega \).
(e) Show that \( \lim_{T \to 0} L_d(e^{j\omega T}) = L(s) \) for a fixed \( s \).
Problem 4 (Control System Design for Overshoot and Settling Time)
(a) Suppose that we have a plant \( P = \frac{(s+3)^2}{(s-1.5)(s+1)(s+2)^2} \left[ \frac{10-5}{s+10} \right]^2 \left[ \frac{10^3}{s+10^5} \right]^4 \). Design a feedback control system such that the closed loop system (1) is stable, (2) exhibits zero steady state error to step commands as well as to step input and output disturbances, (3) exhibits a settling time of approximately \( t_s = 20 \) sec to step commands, (4) exhibits an approximate overshoot of 10\% when a step command is issued, (5) attenuates the impact of high frequency sensor noise on the controls. Support your design with a simple root locus plot.
(b) For your design, specify a suitable sampling frequency and determine the form of an appropriately discretized controller.

Problem 5 (Control System Design for BW and PM)
(a) Nominal Design. Suppose that we have a plant \( P = \frac{s+3}{s(s-1)(s+1)(s+2)} \left[ \frac{5-s}{s+5} \right]^2 \left[ \frac{100}{s+100} \right]^4 \left[ \frac{0.5\Omega_s}{s+0.5\Omega_s} \right]^2 e^{-s(\pi r)} \) where \( \Omega_s = k\omega_g \) and \( k > 0 \). Here, the last two terms of \( P \) represent a second order anti-aliasing filter and a half-sample time delay due to the zero order hold. Suppose \( k = 100 \). Design a feedback control system such that the closed loop system (1) is stable, (2) exhibits zero steady state error to ramp output disturbances \( d_o \), (3) exhibits a unity gain crossover of \( \omega_g = 1 \) rad/sec, (4) exhibits a phase margin \( PM = 60^\circ \), (5) high frequency sensor noise has little impact on the controls, (6) overshoot to a step reference command is minimized.
(b) New Design. How would your design change if \( k = 4 \)? Be specific. Compute it!
(c) For the design in (b), determine the form of an appropriately discretized controller.

Problem 6 (Implementation of a Digital Control Systems)
(a) Sketch a block diagram for a digital control system. (b) Briefly describe each of its components and a reasonable design process. (c) Sketch desirable frequency response shapes for \(|Tr_y|, |S|, |T|, |KS|, |PS|\). Briefly explain each shape.

Problem 7 (Discretization Methods)
Consider the system \( L = \frac{c-s}{(s+a)(s+b)} \) where \( a, b, c > 0 \). Obtain a discretization for \( L \) using the bilinear transformation and the step invariance (zero order hold) methods. Compare the resulting discretizations in terms of poles, zeros, step response.

Problem 8 (Controller Design - EXTRA CREDIT)
Consider the system \( P = \frac{1}{(s-p_1)(s-p_2)} \) where \( p_1, p_2 > 0 \). Show how to design a control system which exhibits zero steady state error to step reference commands with dominant closed loop poles: \(-\sigma_o \pm j\omega_o, -\sigma_1\). Address overshoot due to step reference commands and attenuation of high frequency sensor noise. Show how to solve for the controller coefficients. HINT: Use the PID controller \( K(s) = \frac{g(s^2+bs+c)}{s} \). Form closed loop characteristic equation and solve.