Continuous-Time Modeling of a Digital Control System

Goal:
We wish to obtain a continuous-time model for a digital control system - one that captures the very critical lag properties associated with a digital implementation.
Visualization of a Digital Control System

H - hold device (zero-order hold)
S - sample device
T - clock period

Notes:

1. The above feedback system is a synchronous digital control system. This means that there is a digital clock (not shown above) that synchronizes all blocks on the digital (left hand) side of the loop; i.e. 
   KML, D/A, H on top, A/D, H on bottom, and S.

2. With today's high-level development tools, we need not worry about directly implementing a digital controller Kd at the machine level (ML) - a process that can be very tedious and would generally significantly increase updating and maintenance time.

   It should be noted, however, that a direct machine- or low-level implementation may be necessary if optimized code is essential.
The above digital control system is quite complex. We wish to obtain a simpler mathematical model that is useful for analysis and design.

Toward this end, we begin by making the most common technical simplifications.

Assumptions:  
1. Neglect D/A
2. Neglect A/D
3. Neglect hold associated with A/D.

This cannot be done for high frequency operation where a hold on the A/D input is essential for the A/D to convert accurately. In such a case, the hold may contribute a non-negligible time delay to the feedback loop.

Given the above, we obtain the following simplified closed loop system (CLS):
Suppose \[ K_a = S K H \]

Substituting this into the loop yields the following OLS:

\[ y(s) = (P(s)) (A(s)) H S u \]

\[ U(s) = K(s) \left( \frac{1-e^{-sT}}{s} \right) S \hat{y} \]

\[ \hat{y}(s) = F(s) G(s) Y(s) \]

Assuming \( u \) and \( \hat{y} \) are sufficiently bandlimited, then the infinite sums associated with the samplers may be approximated at low freq. by

\[ \frac{1}{T} \sum U(s+Tn) \approx \frac{1}{T} U(s) \]

\[ \frac{1}{T} \sum \hat{y}(s+Tn) \approx \frac{1}{T} \hat{y}(s) \]

Doing so yields

\[ \mathbf{L}(s) \approx P(s) K(s) A(s) F(s) G(s) \left[ \frac{1-e^{-sT}}{sT} \right]^2 \]

\[ \text{This is an approximation... } L \text{ is really } LT \text{V... due to two holds & samplers} \]
Since 
\[
\frac{1-e^{-j\omega T}}{j\omega T} = e^{-j\omega T/2} \left[ \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{j\omega T/2} \right] \frac{1}{2}
\]
\[
= e^{-j\omega T/2} \left( \frac{j \sin (\omega T/2)}{j (\omega T/2)} \right)
\]
\[
= e^{-j\omega T/2} \left( \frac{\sin (\omega T/2)}{(\omega T/2)} \right)
\]
\[
\approx e^{-j\omega T/2} \quad \text{at low } \omega
\]

This implies that the following continuous-time open loop approximation may be used for control system analysis and design:

\[
L(s) \approx P(s) K(s) A(s) F(s) \ G(s) \ e^{-ST}
\]

After \( K \) is designed, it can be discretized to get \( K_d(z) \).

One can then compare the frequency responses of

\[
L(s) \overset{\text{desired}}{=} P(s) K(s) A(s) F(s) \ G(s) \left[ \frac{1-e^{-ST}}{sT} \right]^2
\]

and

\[
L(s) \overset{\text{actual}}{=} P(s) K_0(e^{sT}) A(s) F(s) \ G(s) \left[ \frac{1-e^{-ST}}{sT} \right]
\]

\[
\text{captures lag due to } P, K, A, F, G \text{ as well as } D
\]