Chapter 2

An Introduction To Laplace Transforms

Many dynamical systems may be modelled or approximated by linear ordinary differential equations with constant coefficients (e.g. aerospace systems, bio-economic systems, chemical systems, electrical systems, mechanical systems). An indispensable tool for analyzing such systems is the so-called unilateral Laplace transform. Basic references on this material are as follows: [52], [68], [72], [103], [165], [212], [214], [247]. Advanced concepts and applications are treated within [71] and [285].

Application to Differential Equations. The utility of Laplace transforms is broad [285]. They can be used to analyze signals. They are also very useful for analyzing systems described by ordinary differential equations with constant coefficients. This is accomplished in two steps.

1. Transform to Laplace $s$-Domain. First, we transform the differential equation to obtain a simple algebraic relationship involving the Laplace variable $s$. The variable $s$, as will be seen, fundamentally represents the differentiation operation $\frac{d}{dt}$ from differential calculus. That is, we make the fundamental association

$$\frac{d}{dt} \leftrightarrow s.$$

The variable $s$, therefore, is sometimes referred to as a differential operator.

2. Transform back to $t$-Domain. After the problem is transformed into the $s$-domain, algebra is typically used to manipulate the resulting $s$-domain expression so that standard Laplace transform tables may be used to “solve” the differential equation by transforming the (manipulated) algebraic expression in $s$ back into the $t$-domain.

In this chapter, the unilateral Laplace transform is defined, elementary transform pairs and properties are examined, and relevant applications are covered. The purpose of the chapter is to show how the Laplace transform may be used to systematically analyze dynamical systems and processes that arise in science and engineering applications. As such, the chapter establishes an essential foundation for the study of dynamical systems and processes.

Historical Note. The Laplace transform is named after the French mathematician, physicist, and astronomer Pierre-Simon Laplace (1749-1827). Although Laplace used the transform in his work on probability theory, the transform was originally discovered by the Swiss mathematician Leonhard Euler (1707-1783). It

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1Most dynamical systems or processes that occur in engineering applications may be approximated by such differential equations. Remember, however, that all models and approximations have limitations!
was Oliver W. Heaviside (1850-1925), an English electrical engineer, however, that exploited complex numbers and Laplace transforms to analyze electrical circuits. He is responsible for the fundamental association \( \frac{d}{dt} \leftrightarrow s \) which permits us to transform a differential equation into an algebraic equation. As such, he is the inventor of the universally accepted transformation-algebra-inverse methodology described within this chapter. Heaviside is also credited with introducing the electrical circuit terms capacitance, inductance, and impedance. As was Charles Dickens, Heaviside was born in the slums of London.

**Recommended Review.** The following review is recommended before continuing.

- **Differential Equations.** It is strongly recommended that the reader review basic concepts on the solution of ordinary differential equations with constant coefficients (e.g. Appendix ??). The reader should specifically review the following concepts: characteristic equation, characteristic roots, and how the latter impacts solutions. Ask yourselves: When does a solution grow? When does it decay? When does it remain constant? When does it oscillate? Such knowledge for first, second, and third order differential equations would provide valuable perspective and establish a useful foundation for this chapter.

- **Signals.** It is also recommended that the reader review Appendix ?? on mathematical models for continuous-time signals. That appendix discusses common test signals. These include: delta distributions (more commonly referred to as impulse functions), step functions, exponentials, sinusoids, ramps, and exponential sinusoids.

- **Complex Arithmetic.** The reader is also strongly advised to examine Appendix ?? for an overview of complex arithmetic. Specifically, see Exercises ??, ??.

This chapter may be skipped by individuals familiar with Laplace transforms and their use for analyzing signals, systems, and ordinary differential equations with constant coefficients.

### 2.1 The Unilateral Laplace Transform: An Introduction

Integral transforms are very important in the study of signals and systems. The *unilateral Laplace transform* is now defined.

**Definition 2.1.1 (Unilateral Laplace Transform)**

The *unilateral Laplace transform* of a function \( f(\cdot) \) is denoted \( F(s) = (\mathcal{L}f)(s) \) and defined as follows:

\[
F(s) = (\mathcal{L}f)(s) \overset{\text{def}}{=} \int_{0^-}^{\infty} f(\tau)e^{-s\tau}d\tau. \tag{2.1}
\]

The set of complex numbers \( s \) for which the Laplace integral converges (i.e. makes sense, is finite) is called the *region of convergence (ROC) of \( F(\cdot) \).*

**Comment 2.1.1 (Delta Distribution and Lower Limit of Laplace Integral)**

The purpose of the minus sign on the lower limit of the Laplace integral is to ensure that unit Dirac delta distributions - denoted by the symbol \( \delta \) - situated at the origin are captured by the Laplace integral. A unit delta distribution (also called unit delta function) can be thought of as a pulse which is infinitely tall, infinitesimally narrow, with unit area, that is situated at the origin, and that is symmetric with respect to the vertical axis (see Appendix ??); i.e.

\[
\int_{0^-}^{0^+} \delta(\tau) \, d\tau = 1 \tag{2.2}
\]

\[
\delta(t) = \delta(-t). \tag{2.3}
\]
The key defining property of a delta distribution is the so-called sifting property:

\[ \int_{t_o}^{t_o+} f(\tau) \delta(\tau - t_o) \, d\tau = f(t_o) \quad (2.4) \]

Delta functions should be viewed as models for tall thin pulses. Such functions, signals, or waveforms are used as test signals in a wide range of applications. It can be shown, for example, that the function \( \delta_\epsilon(t) = \frac{\sin \pi \epsilon t}{\pi t} \) approximates a delta function as \( \epsilon \to 0^+ \). Delta functions are formally discussed in Appendix ??.

**Existence of Unilateral Laplace Transform.** In general, the Laplace transform integral may or may not converge for a given value of \( s \). The following theorem [285, pp. 2] provides sufficient conditions for the Laplace transform of a function \( f \) to exist; i.e. for it to be well defined.

**Theorem 2.1.1 (Existence of Unilateral Laplace Transform: Sufficient Conditions)**

Consider a real valued function \( f : \mathbb{R} \to \mathbb{R} \). Suppose that

- \( f \) is piecewise continuous on every finite interval \([a, b]\).
- \( f \) is of exponential order for \( t \geq 0 \); i.e. Given \( N \geq 0 \), there exists constants \( M \geq 0 \) and \( \sigma_o \geq 0 \) such that

\[ |f(t)| < Me^{\sigma_o t} \text{ for all } t > N. \quad (2.5) \]

Given the above, the unilateral Laplace transform \( F(s) = (L f)(s) \) of \( f \) exists and is well defined for all \( s \) satisfying \( \text{Re } s > \sigma_o \).

The conditions given within the above theorem are satisfied by many functions \( f \) encountered within science and engineering applications. Because the continuity condition can be weakened, it follows that we fundamentally just need \( f \) to grow no faster than some exponential. This condition is satisfied in many important science and engineering applications. It should be emphasized that this is a sufficient condition; not a necessary condition. That is, signals which do not satisfy the above condition may still possess a meaningful Laplace transform. The function \( f(t) = e^{t^2}1(t) \) does not satisfy the exponential order condition. It can be shown that its Laplace transform does not exist for any \( s \).

**Elementary Transform Pairs.** A list of commonly used elementary Laplace transform pairs is provided in Table 2.1. As we progress through the chapter, some of the associations (pairs) given in the table will be validated.

**Laplace Transform Properties.** A comprehensive list of Laplace transform properties is provided in Table 2.2. Properties will be validated as we progress through the chapter.

**Comment 2.1.2 (Use of Laplace Transform Tables)**

Do not be intimidated by Tables 2.1-2.2. Transform pairs and properties will be developed and discussed as we progress through the chapter. After a while, fundamental associations between \( t \)-domain functions \( f \) and associated \( s \)-domain Laplace transforms \( F \) will become second nature and Tables 2.1-2.2 will become useful tools - just like integral tables found in calculus texts. Also, after a while you will get a feel for what are the most useful pairs and properties. To a large extent, that will depend on the problems emphasized by your instructor. You certainly should not try to memorize everything! What you ought to try to do is figure out how the more “complex looking” transform pairs may be obtained from the “simpler looking” transform pairs. You would like to become proficient at this.
Section 2.1: The Unilateral Laplace Transform - An Introduction

<table>
<thead>
<tr>
<th>Table 2.1: Elementary Laplace Transform Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t); \ t \geq 0 )</td>
</tr>
<tr>
<td>1. ( \delta(t) )</td>
</tr>
<tr>
<td>2. ( k )</td>
</tr>
<tr>
<td>3. ( ke^{-at} )</td>
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<tr>
<td>4. ( kt )</td>
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<tr>
<td>5. ( \frac{t^n}{n!}e^{at} )</td>
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<tr>
<td>6. ( \cos(\omega_n t) )</td>
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<tr>
<td>7. ( \sin(\omega_n t) )</td>
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<tr>
<td>8. ( \sin(\omega_n t + \theta) )</td>
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<tr>
<td>9. ( e^{\sigma t} \sin(\omega_n t + \theta) )</td>
</tr>
<tr>
<td>10. ( c \left[ e^{-at} - e^{-bt} \right] )</td>
</tr>
<tr>
<td>11. ( \frac{c}{s-a} \left[ 1 - e^{-at} \right] )</td>
</tr>
<tr>
<td>12. ( \left( c - a \right)e^{-at} - \left( c - b \right)e^{-bt} )</td>
</tr>
<tr>
<td>13. ( 1 - \cos \omega_n t )</td>
</tr>
<tr>
<td>14. ( t \cos \omega_n t )</td>
</tr>
<tr>
<td>15. ( t \sin \omega_n t )</td>
</tr>
<tr>
<td>16. ( \frac{k_n^2}{s^2 + \omega_n^2} e^{-at} + \frac{k_n}{\sqrt{\omega_n^2 + s^2}} \sin \left( \omega_n t - \tan^{-1} \frac{\omega_n}{s} \right) )</td>
</tr>
<tr>
<td>17. ( \frac{k_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \left( \omega_n \sqrt{1 - \zeta^2} t + \tan^{-1} \frac{\omega_n}{s} \right) )</td>
</tr>
<tr>
<td>18. ( 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \left( \omega_n \sqrt{1 - \zeta^2} t + \tan^{-1} \frac{\omega_n}{s} \right) )</td>
</tr>
<tr>
<td>19. ( \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \left( \delta(t - \Delta) \right) )</td>
</tr>
<tr>
<td>20. ( \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \left( \delta(t - \Delta) \right) )</td>
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<tr>
<td>21. ( e^{-a(t-\Delta)} )</td>
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<tr>
<td>22. ( e^{-a(t-\Delta)} )</td>
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</tbody>
</table>
2.2 Elementary Unilateral Laplace Transforms

The following example demonstrates the linearity property of the Laplace transform.

**Example 2.2.1 (Linearity of Laplace Transform)**

Suppose that \( f_1(t) \xrightarrow{\mathcal{L}} F_1(s) \) and \( f_2(t) \xrightarrow{\mathcal{L}} F_2(s) \). Show that

\[
a_1 f_1(t) + a_2 f_2(t) \xrightarrow{\mathcal{L}} a_1 F_1(s) + a_2 F_2(s).
\]  

(2.6)

**Solution:** Let \( g(t) = a_1 f_1(t) + a_2 f_2(t) \) and \( g(t) \xrightarrow{\mathcal{L}} G(s) \). To obtain the result, one proceeds as follows

\[
G(s) \overset{\text{def}}{=} \int_0^\infty g(\tau)e^{-s\tau}d\tau = \int_0^\infty \left[ a_1 f_1(\tau) + a_2 f_2(\tau) \right] e^{-s\tau}d\tau
\]

(2.7)

\[
= a_1 \int_0^\infty f_1(\tau)e^{-s\tau}d\tau + a_2 \int_0^\infty f_2(\tau)e^{-s\tau}d\tau = a_1 f_1(s) + a_2 f_2(s).
\]

(2.8)

**Comment 2.2.1 (The Concept of Linearity)**

The concept of linearity is a very important concept. It arises when studying linear functions, operators, differential equations, or systems.
• The function \( f(x) = 10x \) is a linear function since \( f(a_1x_1 + a_2x_2) = a_1f(x_1) + a_2f(x_2) \). By this definition of linearity, the function \( f(x) = x + 1 \) is NOT linear! (Yes, many texts refer to such a function as being linear. That does not mean they are correct. It means that they are a bit loose in their terminology. A more appropriate term for a function possessing the form \( f(x) = mx + b \) is the term affine function.

• The above example shows that the Laplace transform operator \( \mathcal{L} \) is a linear operator.

• Linearity also arises when studying linear differential equations. We say that the differential equation \( \dot{y} + 5u = 10u \) is linear because if \( y_1 \) and \( y_2 \) are solutions, then \( y = a_1y_1 + a_2y_2 \) is also a solution. In contrast, the differential equation \( \dot{y} + y^2 = u \) is nonlinear. Similar ideas apply to dynamical systems.

Tall thin pulse-like signals are often used as test functions for dynamical systems. Delta distributions are commonly used to model such signals (see Appendix ??).

Example 2.2.2 (Dirac Delta Distribution, Impulse Function)
Show that
\[
\delta(t) \xrightarrow{\mathcal{L}} 1. \tag{2.10}
\]

Solution: The result follows from the sifting property of the delta distribution:
\[
(\mathcal{L}\delta)(s) \overset{\text{def}}{=} \int_0^\infty \delta(\tau)e^{-st} \, d\tau = \int_0^+ \delta(\tau) \, d\tau = 1. \tag{2.11}
\]

Perhaps the best illustration of a delta-like (tall, thin, pulse-like) function is given by the force that a nail experiences when struck by a hammer.

Comment 2.2.2 (Delta Function: Model for Wide-Band Signals)
When evaluated on the imaginary axis, the Laplace transform of a signal measures the frequency or spectral content of the signal. Given this, the above implies that the delta function contains all frequencies. This suggests that \( \delta \) functions should be used to model wide-band signals; i.e. signals with a wide range of sinusoidal frequency components.

• The constant signal \( f_o(t) = 10 \) contains one frequency, namely \( \omega = 0 \) or dc. (The term dc comes from direct current - denoting a current flowing in one direction.)

• The sinusoidal signal \( f_1(t) = \sin t + \sin 2t \) contains two frequency components; i.e. \( \omega = 1 \) and \( \omega = 2 \) rad/sec.

• The signal \( f_2(t) = \sum_{n=20}^{1000} \sin 2\pi nt \) contains many frequency components - over a “wide” range of frequencies.

Here, the term “wide” is being used a bit arbitrarily. Whether a signal is a wide-band signal (containing a wide range of sinusoidal components) or a narrow-band signal (containing a narrow range of sinusoidal components) depends on the application; i.e. what the signal is being used for. If \( f_2 \) represents a disturbance acting on an automobile suspension system, then it is indeed a wide-band signal (with frequencies 20-1000 Hz). If \( f_2 \) represents the the input to a video amplifier, then it is a low frequency signal.

Test Signals Used by Engineers and Small Curious Children. \( \text{It should also be noted that engineers often use impulsive delta-like signals for testing systems (e.g. structures, vehicles, circuits). It should also be noted that small children do the same thing! Have you ever observed a 1 year old becoming familiar with objects in their environment? They show a particular interest in examining how hand held electronic devices, such as remote controls, respond to impulse-like signals.} \)
Step-like signals are also widely used to test dynamical systems (e.g. structures, vehicles, circuits). Consider, for example, the application of a constant force to a car or the application of a constant heat source within a semiconductor furnace.

Example 2.2.3 (Unit Step Function)
Consider the unit step function \(1(t)\) defined as:
\[
1(t) = \begin{cases} 
0 & -\infty < t < 0 \\
1 & 0 \leq t < \infty.
\end{cases}
\]
(see Appendix ??). Show that
\[
1(t) \overset{\mathcal{L}}{\rightarrow} \frac{1}{s}, \quad \text{Re} \ s > 0.
\] (2.12)

Solution: Letting \(f(t) = 1(t)\), it follows that
\[
F(s) = \int_{0^-}^{\infty} e^{-st} \, dt = \left[ \frac{e^{-st}}{-s} \right]_0^\infty = \left[ \frac{e^{-st}}{s} \right]_\infty.
\] (2.13)

Letting \(s = \sigma + j\omega\) in the above exponential yields
\[
F(s) = \frac{e^{-(\sigma+j\omega)t}}{s} \bigg|_{t=\infty}^{0} = \frac{1}{s} \left[ 1 - e^{-\sigma t} e^{j\omega t} \right]_{t=\infty}.
\] (2.14)

From this it follows that the right hand side makes sense (i.e. is finite) if and only if \(\text{Re} \ s > 0\). In such a case, we have
\[
F(s) = \frac{1}{s}
\] (2.15)
where \(\text{Re} \ s > 0\) is the region of convergence of \(F(\cdot)\); i.e. the set of \(s\) values in the complex plane for which the integral
\[
F(s) = \int_{0^-}^{\infty} e^{-st} \, dt
\] (2.16)
makes sense, is given by
\[
\text{ROC}_F = \{ s \in \mathbb{C} \mid \text{Re} \ s > 0 \}.
\] (2.17)

For any other values of \(s\), the integral makes no sense. Try, for example, the values \(s = -1, -1 + j1, j1\). For these values, which lie outside the region of convergence \(\text{ROC}_F\), the integral makes no sense. Also try the values \(s = 1, 1 + j1\). For these values, which lie within \(\text{ROC}_F\), the integral makes sense.

Comment 2.2.3 (Steps and Poles at the Origin, Differential Equation, Transfer Function)
The above example shows that step signals possess a transform with a single pole\(^2\) at \(s = 0\) - the origin of the \(s\)-plane. Single poles at \(s = 0\) are henceforth associated with step signals.

Integration and Differentiation. Since \(1(t) = \int_0^t \delta(\tau) \, d\tau\), it is natural to
associate \(\frac{1}{s}\) with integration and \(s\) with differentiation.

Differential Equation and System Transfer Function. This motivates the associations
\[
y + a_1 y + a_0 y = b_1 \dot{u} + b_0 u \iff (s^2 + a_1 s + a_0)Y(s) = (b_1 s + b_0)U(s) \iff H(s) \overset{\mathcal{L}}{\rightarrow} \frac{Y(s)}{U(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0},
\]
where \((a_i, b_i)\) are constant, \(Y\) is the Laplace transform of \(y\), and \(U\) is the Laplace transform of \(u\). If \(y\) represents the output of a system and \(u\) its input, then the function \(H\) is called the system transfer function or the transfer function associated with the differential equation. More generally,
\[^2\text{A function } F \text{ is said to be rational if it is the ratio of polynomials. Such a function has a pole at } s_0 \text{ if } \lim_{s \to s_0} F(s) = \infty.\]
every real-rational Laplace transform (in principle) can be associated with a differential equation and vice versa.

We will see that it is this association that makes the Laplace transform an ideal tool for studying systems that are described by, or may be approximated by, linear ordinary differential equations with constant coefficients. We will see that the above associations are justified when all initial conditions are zero.

Exercise 2.2.1 (Linearity, Impulse, Step)
(a) Determine the Laplace transform \( F \) of \( f(t) = 5\delta(t) - 61(t) \). Answer: \( F = 5 - \frac{6}{s} = 5 \left[ \frac{s-6}{s} \right] \)
(b) Suppose that \( G(s) = k \left[ \frac{s-\frac{1}{2}}{s} \right] \). Determine \( g \). Hint: \( G(s) = k \left[ 1 - \frac{1}{s} \right] \). Answer: \( g = k [\delta(t) - \frac{1}{s} 1(t)] \)

Exponential functions, as will be seen, are particularly important in the study of linear dynamical systems. Consider, for example, the decay of a car’s initial speed due to aerodynamic drag or temperature decay within a furnace after the furnace is opened to the relatively cold air.

Example 2.2.4 (Exponential Function)
Show that
\[
e^{at}1(t) \leftrightarrow F = \frac{1}{s-a}, \quad Re \ s > Re \ a
\]
where \( a \in \mathbb{C} \) is an arbitrary complex number.

Solution: Letting \( f(t) = e^{at}1(t) \), it follows that
\[
F(s) = \int_0^\infty e^{at}e^{-st} \, dt = \int_0^\infty e^{-(s-a)t} \, dt = \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \left[ \frac{1}{s-a} \right]_0^\infty.
\]
Letting \( s = \sigma + j\omega \) in the above exponential yields
\[
F(s) = \left[ \frac{e^{-(\sigma+j\omega-a)t}}{s-a} \right]_0^\infty = \frac{1}{s-a} \left[ 1 - e^{-(\sigma-a)\infty} e^{j\omega\tau} \right]_{\tau \to \infty}.
\]
From this, it follows that the right hand side makes sense (i.e. is finite) if and only if \( Re \ s = \sigma > Re \ a \). It thus follows that
\[
F(s) = \frac{1}{s-a}
\]
where \( Re \ s > Re \ a \) is the region of convergence of \( F(\cdot) \); i.e. the set of \( s \) values in the complex plane for which the integral
\[
F(s) = \int_0^\infty e^{-(s-a)t} \, dt
\]
makes sense is given by
\[
ROC_F = \{ s \in \mathbb{C} \mid Re \ s > Re \ a \}.
\]
For any other values of \( s \), the integral makes no sense.

Comment 2.2.4 (Exponentials and Single Poles)
The above example shows that exponential signals with a parameter \( a \) (as above) have transforms with a
single pole at \( s = a \). A single pole at \( s = a \) is henceforth associated with an exponential signal.

Stability, Time Constant, Settling Time. The exponential \( f(t) = e^{at}1(t) \) is said to be stable if \( a < 0 \); i.e., if its associated pole at \( s = a \) lies in the open left half \( s\)-plane \((\text{Re} \ s < 0)\). In such a case, the exponential decays to zero as \( t \) approaches \( \infty \). When \( a < 0 \), the rate of decay of the exponential \( e^{-|a|t} \) is governed by the size of \( |a| \). In such a case we associate with the exponential, a time constant

\[
\tau \overset{\text{def}}{=} \frac{1}{|a|} \tag{2.24}
\]

and a settling time

\[
t_s \overset{\text{def}}{=} 5\tau. \tag{2.25}
\]

The idea here is that by \( t = t_s \), the exponential is nearly zero since \( e^{-|a|t_s} = e^{-5} = 0.0067379 \approx 0 \). This “5 time constant settling time convention” is widely used in scientific and engineering applications.

Car Application. Suppose a car with an initial speed \( v_0 \) takes approximately 10 seconds to come to rest as a result of aerodynamic drag; i.e., no brakes are applied. What is the associated (approximate) settling time \( t_s \), time constant \( \tau \), and pole? Answers: \( t_s = 10 \), \( \tau = \frac{10}{5} = \frac{10}{5} = 2 \), pole at \( s = -\frac{1}{\tau} = -\frac{1}{2} \).

Performance Specifications. It should be noted that the concepts of settling time and time constant (and hence pole location) are often used as measures of performance or to state performance specifications for dynamical systems. This will be revisited within Section ??.

Exercise 2.2.2 (Linearity, Steps, and Exponentials)
A hypersonic (X-43A-like waverider) vehicle with an air-breathing scram jet propulsion system is cruising at Mach 7 at an altitude of 110,000 feet (20.83 miles). Its velocity profile is given by \( v(t) = 7 + 3(1 - e^{-t}) \) where \( t \) is measured in seconds. Determine the Laplace transform \( \mathcal{L}\{v\} \). What is the associated time constant? Setting time? Answer: \( \mathcal{L}\{v\} = \frac{10}{s} - \frac{3}{s+1} = \frac{7s+10}{s(s+1)} \) \((\text{Re} \ s > \max\{0, -1\} = 0)\), \( \tau = 1 \text{ sec} \), \( t_s = 5 \text{ sec} \).

Exercise 2.2.3 (Linearity, Impulse, Step, Exponential)
Suppose that \( G(s) = k \left[ \frac{e^{-z}}{s+p} \right] \). Determine \( g \). Hint: \( G(s) = k \left[ \frac{s+p-(p+1)}{s+p} \right] = k \left[ 1 - \frac{(p+1)}{s+p} \right] \). Answer: \( g(t) = k[\delta(t) - (p+z)e^{-pt}] \).

Exercise 2.2.4 (Linearity, Exponential, Geometric Series, Hyperbolic)
(a) Suppose that \( f(t) = \left( \frac{\beta}{\beta - e^{-pt}} \right) 1(t) \) where \( p > 0 \), \( \beta > 1 \). Show that \( f(t) = \left( \frac{1}{\frac{1}{2}e^{-pt}} \right) 1(t) = \sum_{n=0}^{\infty} \left( \frac{-\pi_n}{\beta} \right)^n 1(t) \).

Hint: \( \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \) when \( |r| < 1 \). (b) Show that \( F(s) = \sum_{n=0}^{\infty} \frac{\beta^n}{s+p} = \frac{1}{s} + \frac{\frac{1}{2}}{s+p} + \frac{\frac{1}{4}}{s+2p} + \cdots \), \( \text{Re} \ s > 0 \).
(c) Consider \( g(t) = \sinh at \) where \( a > 0 \). Show that \( G(s) = \frac{a}{s^2 - a^2}, \text{Re} \ s > a \).
(d) Consider \( h(t) = \cosh at \) where \( a > 0 \). Show that \( H(s) = \frac{a}{s^2 - a^2}, \text{Re} \ s > a \).

Sinusoidal signals are arguably the most commonly used test signals. They are particularly useful in the study of linear dynamical systems. They can, for example, be used to model low frequency wind disturbances acting on an aircraft or high frequency sensor noise associated with a gyroscopic sensor. The altitude control system of an unmanned air vehicle (UAV) may receive a low frequency sinusoidal-like reference command, while operating in terrain following mode, as the vehicle traverses a mountain range.

Example 2.2.5 (Sinusoidal Functions)
Show that
Section 2.2: Elementary Unilateral Laplace Transforms

\[
\sin \omega_o t \ 1(t) \quad \mathcal{L} \to \frac{\omega_o}{s^2 + \omega_o^2} \quad \text{Re } s > 0 \quad (2.26)
\]

\[
\cos \omega_o t \ 1(t) \quad \mathcal{L} \to \frac{s}{s^2 + \omega_o^2} \quad \text{Re } s > 0 \quad (2.27)
\]

\[
\sin(\omega_o t + \theta) \ 1(t) \quad \mathcal{L} \to \frac{(\sin \theta)s + \omega_o \cos \theta}{s^2 + \omega_o^2} \quad \text{Re } s > 0 \quad (2.28)
\]

\[
\cos(\omega_o t + \theta) \ 1(t) \quad \mathcal{L} \to \frac{(\cos \theta)s - \omega_o \sin \theta}{s^2 + \omega_o^2} \quad \text{Re } s > 0 \quad (2.29)
\]

Also show that

\[2|C| \cos(\omega_o t + \angle) \ 1(t) \quad \mathcal{L} \to \frac{C}{s - j\omega_o} + \star \quad \text{Re } s > 0 \quad (2.30)\]

where \(C = |C|e^{j\angle} \) and \(\star\) denotes the complex conjugate of the previous term. This latter result is used extensively in finding inverse transforms of real-rational functions with purely imaginary poles.

**Solution:** The first two results follow immediately from the Laplace transform of an exponential and application of Euler’s formulae

\[
sin \omega_o t \ 1(t) = \left[ e^{j\omega_o t} - e^{-j\omega_o t} \right] / 2j \quad (2.31)
\]

\[
\cos \omega_o t \ 1(t) = \left[ e^{j\omega_o t} + e^{-j\omega_o t} \right] / 2 \quad (2.32)
\]

The third result follows immediately from the following trigonometric identity

\[
\sin(\omega_o t + \theta) \ 1(t) = \sin \omega_o t \cos \theta + \cos \omega_o t \sin \theta \quad (2.33)
\]

linearity and application of the above transform pairs for \(\cos \omega_o t \ 1(t)\) and \(\sin \omega_o t \ 1(t)\). The fourth result follows immediately from the following trigonometric identity

\[
\cos(\omega_o t + \theta) \ 1(t) = \cos \omega_o t \cos \theta - \sin \omega_o t \sin \theta \quad (2.34)
\]

linearity and application of the above transform pairs for \(\cos \omega_o t \ 1(t)\) and \(\sin \omega_o t \ 1(t)\).

Finally, we consider

\[2|C| \cos(\omega_o t + \angle) \ 1(t) = |C|e^{j(\omega_o t + \angle)} + \star = |C|e^{j\angle}e^{j\omega_o t} + \star = Ce^{j\omega_o t} + \star. \quad (2.35)\]

Taking transforms yields the desired final result.

**Comment 2.2.5 (Sinusoids and Complex Conjugate Poles on the Imaginary Axis)**

The above example shows that sinusoidal signals with an angular frequency \(\omega_o\) have transforms with complex conjugate imaginary poles at \(s = \pm j\omega_o\). Complex conjugate poles on the imaginary axis are henceforth associated with sinusoidal signals.

**Exercise 2.2.5 (Linearity, Steps, Sinusoids)**

(a) The velocity (Mach) profile \(v\) of a hypersonic vehicle experiencing a longitudinal sinusoidal speed disturbance has transform \(V(s) = \frac{10}{s} + \frac{1}{s^2 + (\frac{10}{10})^2} \). Determine the angular frequency \(\omega_o\) of the disturbance, the associated period \(T\), and the Mach profile \(v\). Answer: \(\omega_o = \frac{10}{10}, T = \frac{2\pi}{\omega_o} = 20\ \text{sec}, v = 10 + \sin(\frac{10}{10}t + 30\degree)\).

(b) Suppose that \(b(t) = \sum_{n=0}^{\infty} a_n \cos n\omega_o t + b_n \sin n\omega_o t \) \(1(t)\) where \(\omega_o > 0\). (Note: This is a Fourier series expansion for \(h\) on \(t \geq 0\) - see Section ??.) Show that \(H(s) = \sum_{n=0}^{\infty} \frac{a_n s^2 + b_n s}{s^2 + (n\omega_o)^2}, \text{Re } s > 0\). Note that \(H\) has purely imaginary complex conjugate poles at each of the harmonic frequencies \(s = \pm jn\omega_o\).
Example 2.2.6 (Multiplication by Exponentials and Frequency Shifting Property)

Show that

\[ e^{at} f(t) \leftrightarrow L \rightarrow F(s - a) \]  \hspace{1cm} (2.36)

where \( a \in \mathbb{C} \) is an arbitrary complex number.

Solution: Defining \( g(t) \overset{\text{def}}{=} e^{at} f(t) \) yields

\[ G(s) = (\mathcal{L}g)(s) \overset{\text{def}}{=} \int_{0^-}^{\infty} e^{at} f(\tau) e^{-s\tau} d\tau = \int_{0^-}^{\infty} f(\tau) e^{-(s-a)\tau} d\tau = F(s - a). \]  \hspace{1cm} (2.37)

Comment 2.2.6 (Multiplication by Exponentials and Frequency Shifting Property)

Example 2.2.6 shows that multiplication of a signal \( f \) by an exponential \( e^{at} \) results in a shift in the \( s \)-domain; i.e. \( F(s-a) \). If \( a = j\omega_o \) is purely imaginary, then the function \( F(j\omega) \) - often referred to as a frequency response function - is shifted by an amount \( \omega_o \); i.e. we obtain \( F(j(\omega - \omega_o)) \).

The following example addresses general exponential sinusoids. A growing (unstable) exponential sinusoidal motion is typically associated with the natural (open loop) dynamics of a hovering helicopter. For this reason, helicopter pilots typically require the assistance of stability augmentation system (SAS) or automatic flight control system (AFCS). Decaying (stable) exponential sinusoids arise, for example, when the structural bending modes of an air-to-air missile are excited by a high gee acceleration command (e.g. 60 gees) from the guidance system.

Example 2.2.7 (Exponential Sinusoid: Frequency Shifting Properties)

Show that

\[ e^{\sigma_o t} \sin(\omega_o t + \theta) 1(t) \leftrightarrow L \rightarrow \frac{\sin \theta(s - \sigma_o) + \omega_o \cos \theta}{s^2 - 2\sigma_o s + \sigma_o^2 + \omega_o^2} \hspace{1cm} \text{Re } s > \sigma_o \]  \hspace{1cm} (2.38)

\[ e^{\sigma_o t} \cos(\omega_o t + \theta) 1(t) \leftrightarrow L \rightarrow \frac{\cos \theta(s - \sigma_o) - \omega_o \sin \theta}{s^2 - 2\sigma_o s + \sigma_o^2 + \omega_o^2} \hspace{1cm} \text{Re } s > \sigma_o. \]  \hspace{1cm} (2.39)

Also show that

\[ 2|C|e^{\sigma_o t} \cos(\omega_o t + \angle C) 1(t) \leftrightarrow L \rightarrow \frac{C}{s - \sigma_o - j\omega_o} + * \hspace{1cm} \text{Re } s > \sigma_o \]  \hspace{1cm} (2.40)

where \( C = |C|e^{j\angle C} \) and * denotes the complex conjugate of the previous term. This latter result is used extensively in finding inverse transforms of real-rational functions with complex (conjugate) poles.

Solution: The result follows immediately from the transform pairs given in equations 2.28, 2.29, 2.30 and 2.36.

Comment 2.2.7 (Exponential Sinusoids and Complex Conjugate Poles)

The above exercise shows that exponential sinusoidal signals with parameters \( \sigma_o, \omega_o \) have transforms with complex conjugate poles at \( s = \sigma_o \pm j\omega_o \). Complex conjugate poles at \( s = \sigma_o \pm j\omega_o \) are henceforth associated with exponential sinusoidal signals.

Stability, Time Constant, Settling Time. The above exponential sinusoids are said to be stable if \( \sigma_o < 0 \); i.e. if their associated poles \( s = \sigma_o \pm j\omega_o \) lie in the open left half \( s \)-plane (\( \text{Re } s < 0 \)). In such a case, the exponential sinusoid decays to zero as \( t \) approaches \( \infty \). When \( \sigma_o < 0 \), the rate of decay of the exponential sinusoid is governed by its exponential envelope: \( e^{-|\sigma_o|t} \). In such a case, we associate with the exponential sinusoid a time constant

\[ \tau \overset{\text{def}}{=} \frac{1}{|\sigma_o|} \]  \hspace{1cm} (2.41)
and a settling time
\[ t_s \overset{\text{def}}{=} 5\tau. \] (2.42)

The idea here is that by \( t = t_s \), the exponential sinusoid is nearly zero since \( e^{-|\sigma|t_s} = e^{-5} = 0.0067379 \approx 0 \). This “5 time constant settling time convention” is widely used in scientific and engineering applications.

**Exercise 2.2.6 (Linearity, Step, Exponential Sinusoid)**

The Mach profile of a hypersonic vehicle undergoing a longitudinal speed disturbance has transform \( V(s) = \frac{12}{s} + 3 \left[ \frac{1}{(s+0.05)^2 + \left( \frac{\pi}{5} \right)^2} \right] \). Determine the time constant, settling time, angular frequency, and period of the disturbance. Also determine the Mach (velocity) profile \( v \) and the steady-state speed \( v_{ss} \). Answers: \( \tau = \frac{1}{0.05} = 20 \) sec, \( t_s = 5\tau = 100 \) sec, \( \omega_0 = \frac{\pi}{5} \), \( T = \frac{2\pi}{\omega_0} = 10 \) sec, \( v(t) = 12 + 3e^{-0.05t} \sin \frac{\pi}{5}t \), \( v_{ss} = \lim_{t \to \infty} v(t) = 12 \) (Mach). How long does it take to reach \( v_{ss} \) (approximately)? Answer: Approximately \( t_s = 5\tau = 100 \) sec.

**Exercise 2.2.7 (Linearity, Exponential Sinusoids: Flexible Modes)**

(a) Suppose that \( h(t) = \sum_{n=0}^{\infty} [a_n e^{-\sigma_n t} \cos \omega_d t + b_n e^{-\sigma_n t} \sin \omega_d t] 1(t) \) where \( \sigma_n, \omega_d > 0 \). Show that the associated Laplace transform is given by \( H(s) = \sum_{n=0}^{\infty} \frac{a_n (s+\sigma_n)^2 + b_n \omega_d^2}{(s+\sigma_n)^2 + \omega_d^2} \). Re \( s > -\min_{n=0,1,\ldots} \sigma_n \). Note that \( F \) has complex conjugate poles at each of the complex frequencies \( s = -\sigma_n \pm j\omega_d \). Physically, \( h \) can represent the impulse response of a flexible structure, while \( H \) denotes its so-called transfer function.

(b) Consider \( f(t) = [a_1 e^{-\sigma_1 t} \sin \omega_1 t + a_2 e^{-\sigma_2 t} \sin \omega_2 t] 1(t) \) with \( \sigma_1, \sigma_2, \omega_1 > 0 \). Show that the associated Laplace transform is given by \( F(s) = \frac{(a_1 \omega_1 + a_2 \omega_2)^2 + 2[a_1 \omega_1 \sigma_1 + a_2 \omega_2 \sigma_2 + \sigma_1 \sigma_2 (\omega_1^2 + \omega_2^2)]}{(s+\sigma_1)^2 + \omega_1^2} \).

- **Equal Rates of Decay.** Suppose \( \sigma_1 = \sigma_2 = \sigma \). Show that \( F(s) = [a_1 \omega_1 + a_2 \omega_2] \left[ \frac{s^2 + 2\sigma s + \sigma^2 + \omega^2}{s^2 + 2\sigma s + \omega^2} \right] \) where \( \omega_0^2 = \left( \frac{a_2 \omega_2}{a_1 \omega_1 + a_2 \omega_2} \right) \omega_1^2 + \left( \frac{a_1 \omega_1}{a_1 \omega_1 + a_2 \omega_2} \right) \omega_2^2 \). If we let \( \rho = \left( \frac{a_2 \omega_2}{a_1 \omega_1 + a_2 \omega_2} \right) \), then \( \omega_0^2 = \rho \omega_1^2 + (1-\rho) \omega_2^2 \). Since \( \rho \in (0,1) \), we say that \( \omega_0^2 \) is a convex combination of \( \omega_1^2 \) and \( \omega_2^2 \). Assuming that \( \omega_1^2 < \omega_2^2 \), show that this implies that \( \omega_0^2 < \omega_1^2 < \omega_2^2 \). Given this, it follows that \( F \) has zeros at \( s = -\sigma \pm j\omega_c \) where \( \omega_1 < \omega_c < \omega_2 \); i.e. on the vertical line joining the poles \( s = -\sigma \pm j\omega_1 \) and \( s = -\sigma \pm j\omega_2 \).

- **Equal Frequencies.** Suppose \( \omega_1 = \omega_2 = \omega \). Show that \( F(s) = \omega [a_1 + a_2] \left[ \frac{s^2 + 2\sigma s + \sigma^2 + \omega^2}{(s+\sigma_1)^2 + \omega^2} \right] \) where \( \rho = \left( \frac{a_2}{a_1 + a_2} \right) \in (0,1) \), \( \sigma_1 = \rho \sigma_1 + (1-\rho) \sigma_2 \), \( \omega_c^2 = \omega^2 + \rho \omega_1^2 + (1-\rho) \omega_2^2 = \omega^2 + \sigma_1^2 + \rho(1-\rho) |\sigma_2 - \sigma_1|^2 \). In such a case, the zeros of \( F \) are at \( s = -\sigma_c \pm j\omega_c \) where \( \sigma_c \) is a convex combination of \( \sigma_1 \) and \( \sigma_2 \) and \( \omega_c = \sqrt{\omega^2 + \sigma_1^2 + \rho(1-\rho) |\sigma_2 - \sigma_1|^2} \).

The above is useful when studying flexible structures; e.g. spacecraft with flexible appendages.

The following exercise addresses t-domain scaling.

**Exercise 2.2.8 (Scaling Property)**

Show that
\[ f \left( \frac{t}{a} \right) \overset{\text{def}}{\leftrightarrow} aF(as) \] (2.43)

for any real constant \( a \neq 0 \). Hint: Let \( y(t) = f \left( \frac{t}{a} \right) \). \( Y(s) = \int_0^\infty y(\tau)e^{-s\tau} d\tau = \int_0^\infty f \left( \frac{\tau}{a} \right) e^{-s\tau} d\tau \). Let \( u = \frac{\tau}{a} \) and \( du = \frac{d\tau}{a} \). We then have \( \tau = au \) and \( d\tau = adu \).

The above exercise shows that scaling in time results in spectrum amplitude scaling as well as frequency scaling. If \( a \in (0,1) \), the signal \( f_1(t) = f \left( \frac{t}{a} \right) \) is a time-compression or “fast playback” of \( f(t) \). The spectrum \( F_1(s) = aF(as) \) is obtained by dilating \( F \) in frequency (i.e. producing a wider version \( F(as) \) of \( F \)) and scaling the result by \( a \). Hence,
A consequence of the above example is the following:

**Comment 2.2.8 (Time Multiplication Results In Repeated Poles)**

If the transform \( F \) of \( f \) has a single pole at \( s_o \), then the transform \(- \frac{d}{ds} F(s)\) of the signal \( tf \) will have a repeated (double) pole at \( s_o \).

Why is this? If \( F \) has a single pole at \( s = s_o \), then it has the form \( F(s) = \frac{1}{s-s_o} G(s) \) where \( G \) is some \( s \)-domain function such that \( G(s_o) \neq 0 \). Differentiating \( F \) shows that the transform \(- \frac{d}{ds} F(s)\) of \( tf \) will have a double pole at \( s = s_o \).

Similar statements apply to time dilation:

**dilation in time (slow play back) or \( a \in (1, \infty) \) corresponds to compression (bandwidth reduction) in frequency and spectral amplitude dilation.**

**Exercise 2.2.9 (Bandwidth Reduction: Time Dilation, Frequency Compression)**

The response of a commercial aircraft altitude climb control system to a 1000 foot altitude command is given by \( y_1(t) = 1000(1-e^{-t}) \). When elevator degradation is detected by the on-board health monitoring system, the control system bandwidth (or response time) is reduced so that the new step response is \( y_2 = 1000(1-e^{-0.5t}) \). For each case, determine the vehicle settling time. Determine \( Y_1 \) and \( Y_2 \). How is \( Y_2 \) related to \( Y_1 \) ? Answers: \( t_s = 5 \) sec, \( t_s = 10 \) sec. \( Y_2 \) is a dilation (slowed down version) of \( Y_1 \). \( Y_1 = 1000 \left[ \frac{1}{s(s+5)} \right] \) \( Y_2 = 1000 \left[ \frac{0.5}{s(s+0.5)} \right] \), \( Y_2(s) = 2Y_1(2s) \). Note that \( Y_1 \) has been increased in amplitude and compressed in frequency. We say that \( Y_2 \) and \( Y_2 \) have a lower bandwidth or are slower than \( Y_1 \) and \( Y_1 \). Effectively, the health management system has lowered the overall control system bandwidth in order to place less stress on the degraded elevator. Future health management systems will perform such functions in order to increase overall system reliability and safety.

In many applications, ramp-like signals arise. For example, consider commanding the angular displacement of a motor shaft in order to horizontally transport a set of wafers on a manufacturing line. In such a case, the motor shaft angle intentionally grows with time in a ramp-like fashion; e.g. \( \theta(t) = 10t(1) \). The following example considers more general ramp-like signals.

**Example 2.2.8 (Time Multiplication and Frequency Differentiation Property)**

Show that

\[
 tf(t) \overset{\mathcal{L}}{\longleftrightarrow} - \frac{d}{ds} F(s). \tag{2.44}
\]

**Solution:** Defining \( g(t) \overset{\text{def}}{=} tf(t) \) yields

\[
 G(s) = (\mathcal{L}g)(s) \overset{\text{def}}{=} \int_{0-}^{\infty} tf(t)e^{-st} dt = \int_{0-}^{\infty} f(\tau) \{\tau e^{-st}\} d\tau = \int_{0-}^{\infty} f(\tau) \left\{ \frac{d}{ds} \left[ -e^{-st} \right] \right\} d\tau \tag{2.45}
\]

\[
 = \int_{0-}^{\infty} f(\tau) \left\{ \frac{d}{ds} \left[ e^{-st} \right] \right\} d\tau = - \frac{d}{ds} \int_{0-}^{\infty} f(\tau) e^{-st} d\tau = - \frac{d}{ds} F(s) \tag{2.46}
\]

The above example shows that multiplication by \( t \) in the \( t \)-domain corresponds to differentiation in the \( s \)-domain.

**Comment 2.2.8 (Time Multiplication Results In Repeated Poles)**

A consequence of the above example is the following:

**If** the transform \( F \) of \( f \) has a single pole at \( s_o \), then the transform \(- \frac{d}{ds} F(s)\) of the signal \( tf \) will have a repeated (double) pole at \( s_o \).

Why is this? If \( F \) has a single pole at \( s = s_o \), then it has the form \( F(s) = \frac{1}{s-s_o} G(s) \) where \( G \) is some \( s \)-domain function such that \( G(s_o) \neq 0 \). Differentiating \( F \) shows that the transform \(- \frac{d}{ds} F(s)\) of \( tf \) will have a double pole at \( s = s_o \).
Section 2.2: Elementary Unilateral Laplace Transforms

The following example considers steps, ramps, parabolas, and higher-order powers of \( t \).

**Example 2.2.9 (Powers of \( t \))**

Show that

\[
\frac{t^n}{n!} 1(t) \xleftarrow{\mathcal{L}} \frac{1}{s^{n+1}} \quad \text{for } n = 0, 1, 2, \ldots
\]  

(2.47)

**Solution:** This result follows by induction. First we note that since

\[
1(t) \xleftarrow{\mathcal{L}} \frac{1}{s}
\]  

(2.48)

the result holds for \( n = 0 \). To complete our inductive proof, we assume that the result holds for \( n - 1 \) and prove that it holds for \( n \). If it holds for \( n - 1 \), then it follows that

\[
\frac{t^{n-1}}{(n-1)!} 1(t) \xleftarrow{\mathcal{L}} \frac{1}{s^n}.
\]  

(2.49)

Applying the time multiplication result given in Equation 2.44 yields

\[
\frac{t^n}{(n-1)!} 1(t) \xleftarrow{\mathcal{L}} - \frac{d}{ds} \left[ \frac{1}{s^n} \right] = \frac{n}{s^{n+1}},
\]  

(2.50)

from which the result immediately follows by dividing each side by \( n \).

**Comment 2.2.9 (Ramps and Double Poles at the Origin)**

Example 2.2.9 illustrates the basic fact that multiplication by \( t \) in the \( t \)-domain results in repeated poles in the \( s \)-domain. It specifically shows that while the transform for a unit step \( 1(t) \) has a single pole at \( s = 0 \), the transform of a unit ramp \( t 1(t) \) has a double pole at \( s = 0 \). Double poles at \( s = 0 \) in \( s \)-domain functions are henceforth associated with ramps in the \( t \)-domain. By similar reasoning, the unit parabola \( \frac{t^2}{2!} 1(t) \) is associated with \( \frac{1}{s^3} \) and hence a triple pole at the origin.

The following example addresses time differentiation of functions. As such, it will permit us to directly solve linear ordinary differential equations with constant coefficients. This is very important because many dynamical systems can be approximated by such equations (see Problem 1.6.11, page 40). The position of a car, for example, can be approximated by a second order differential equation; its speed by a first order differential equation. The temperature within a furnace is often approximated by a first order differential equation.

**Example 2.2.10 (Time Differentiation Property)**

Show that

\[
\frac{d}{dt} f(t) \xleftarrow{\mathcal{L}} s F(s) - f(0^-)
\]  

(2.51)

\[
\frac{d^2}{dt^2} f(t) \xleftarrow{\mathcal{L}} s^2 F(s) - s f(0^-) - \dot{f}(0^-)
\]  

(2.52)

\[
\frac{d^3}{dt^3} f(t) \xleftarrow{\mathcal{L}} s^3 F(s) - s^2 f(0^-) - 2 s \dot{f}(0^-) - \ddot{f}(0^-)
\]  

(2.53)

**Solutions:** Defining \( g(t) \overset{\text{def}}{=} \dot{f}(t) \) and integrating by parts yields

\[
G(s) = (\mathcal{L}g)(s) \overset{\text{def}}{=} \int_0^\infty f(\tau) e^{-st} d\tau = \left[ u v \right]_{t=0}^{t=\infty} \quad u = e^{-st} \quad \dot{v} = \dot{f}
\]  

(2.54)

\[
= \left. e^{-st} f(t) \right|_0^\infty - \int_0^\infty f(\tau) \left[ -s e^{-st} \right] d\tau = \lim_{t \to \infty} e^{-st} f(t) - f(0^-) + sF(s)
\]  

(2.55)
Chapter 2: An Introduction To Laplace Transforms

The first result then follows since
\[ \lim_{t \to \infty} e^{-st} f(t) = 0 \]  
for any \( s \) which lies within the region of convergence of \( F \). To obtain the second result, one applies the first result to \( \frac{d^2}{dt^2} f(t) = \frac{d}{dt} \dot{f}(t) = \frac{d}{dt} g(t) \) where \( g(t) = \dot{f}(t) \). Doing so yields \( G(s) = sF(s) - f(0^-) \) and
\[ \frac{d^2}{dt^2} f(t) = \frac{d}{dt} g(t) \leftrightarrow sG(s) - g(0^-) = s[f(s) - f(0^-)] - \dot{f}(0^-) = s^2F(s) - sf(0^-) - \ddot{f}(0^-) \]  
(2.57)

The last result follows similarly, by applying the first result to \( \frac{d^2}{dt^2} f(t) = \frac{d}{dt} \ddot{f}(t) \).

Comment 2.2.10 (s and Differentiation)

Example 2.2.10 shows that differentiation in the \( t \)-domain corresponds to multiplication by \( s \) in the \( s \)-domain. Given this, it is natural to refer to the variable \( s \) as a differentiation operator. This justifies our associating \( s \) with differentiation. Below, it is shown how the results from within Example 2.2.10 can be used to solve linear ordinary differential equations with constant coefficients. As such, Example 2.2.10 contains fundamentally important Laplace transform relationships.

Example 2.2.11 (Application of Derivative Property To a Discontinuous Function)

Consider the discontinuous function
\[ f(t) \overset{\text{def}}{=} \begin{cases} e^{-t}1(t) & 0 \leq t < \infty \\ -1 & -\infty < t < 0 \end{cases} \]  
(2.58)

Sketch the function \( f \). Note that \( f \) possesses a discontinuity of size 2 at \( t = 0 \). Differentiating \( f \) directly yields
\[ \dot{f}(t) = 2\delta(t) - e^{-t}1(t) \]  
(2.59)

where the \( 2\delta(t) \) term is a result of the discontinuity of size 2 at \( t = 0 \). We would like to obtain the above derivative result using the derivative property of the Laplace transform. Noting that \( F(s) = \frac{1}{s+1} \), it follows from Example 2.2.10 that the transform of \( \dot{f} \) is given by
\[ (\mathcal{L}\dot{f})(s) = sF(s) - f(0^-) = s \left( \frac{1}{s+1} \right) - (-1) = \frac{s+1-1}{s+1} + 1 = 2 - \frac{1}{s+1}. \]  
(2.60)

The inverse transform of this yields the desired result.

Exercise 2.2.10 (First Order Ordinary Differential Equation: Impulse Response)

Solve the following linear ordinary differential equation for \( y \)
\[ y + 2y = 6 u(t) \quad \quad y(0^-) = 0 \]  
(2.61)

for \( t > 0 \) with \( u(t) = \delta(t) \). Answer: Using the results from Example 2.2.10, we obtain \( Y(s) = H(s)U(s) = \frac{6}{s+2} \), \( y(t) = h(t) = \mathcal{L}^{-1}(H) = 6e^{-2t}\delta(t) \). Note that \( Y = H \) is the associated transfer function, while \( y = h \) is the associated impulse response. Note: time constant is \( \tau = \frac{1}{2} \), settling time is \( t_s = 5\tau = 2.5 \).

Exercise 2.2.11 (Kinematics for Free Falling Body, Satellite Translational Dynamics)

Kinematics of Free Falling Body. Consider a free falling body with negligible drag. Taking downward as the positive direction, recall that the associated differential equation is given by \( a = \ddot{x} = \ddot{x} = u \) where \( u = g \) is the acceleration due to gravity, \( x \) denotes vertical displacement, and \( \dot{v} \) denotes vertical velocity. Let \( \dot{v} \) and \( x_o \) denote the initial velocity and displacement, respectively. (a) Determine the transfer functions \( H_u \) from \( u \) to \( v \) and \( H_{ux} \) from \( u \) to \( x \). Answers: \( \dot{v} = u \) implies that \( H_u(s) \overset{\text{def}}{=} \frac{\dot{v}}{u} \big|_{\text{zero initial conditions}} = \frac{1}{s} \). This emphasizes that \( \dot{v} \) is the integral of acceleration \( u = a \). Similarly, \( \ddot{x} = u \) implies that \( H_{ux}(s) \overset{\text{def}}{=} \frac{\ddot{x}}{u} \big|_{\text{zero initial conditions}} = \frac{1}{s^2} \).
This emphasizes that \( x \) is the double integral of acceleration \( u = a \). (b) Assuming zero initial conditions, determine \( v \) and \( x \). Answers: \( A = U = \frac{x}{2}, a = g, V = \frac{U}{s} = \frac{a}{2}, v = gt, X = \frac{U}{s^2} = \frac{g}{2}, x = \frac{1}{2}gt^2 \). (c) Repeat (b) assuming non-zero initial conditions. Answers: Using the results from Example 2.2.10, we obtain:

\[
\begin{align*}
\dot{v} &= u, sV - v_o = U, U = \frac{x}{2}, V = \frac{v_o}{s} + \frac{U}{s^2} = \frac{v_o}{s} + \frac{a}{2}, v = gt + v_o, \\
\ddot{x} &= u, s^2 X - sx_o - v_o = U, X = \frac{x_o}{s} + \frac{v_o}{s^2} + \frac{U}{s^3} = \frac{x_o}{s} + \frac{v_o}{s^2} + \frac{g}{2}, x = \frac{1}{2}gt^2 + v_o t + x_o.
\end{align*}
\]

Satellite Translational Dynamics. Now consider a satellite with mass \( m \) and force \( u \) acting on it.

(d) Show that the associated differential equation is \( m \ddot{x} = m \dot{v} = u \) where \( v \) and \( x \) denote velocity and displacement, respectively. (e) Show that the associated transfer functions are \( H_{uu}(s) \define \frac{V}{U} \left. \right|_{\text{zero initial conditions}} = \frac{1}{s} \) and \( H_{ux}(s) \define \frac{X}{U} \left. \right|_{\text{zero initial conditions}} = \frac{1}{s^2} \). We say that the satellite is first order in \( v \) and second order in \( x \).

(f) Determine \( v \) and \( x \) when a constant force \( u = F_o \) is applied at \( t = 0 \) and initial conditions are non-zero.

Answers: \( a = \frac{F_o}{m}, v = at + v_o, x = \frac{1}{2}at^2 + v_o t + x_o \). Note: The rotational dynamics for a satellite are identical in structure with \( u \) representing an applied torque, \( m \) replaced with moment of inertia \( I \) about the axis of rotation, \( v \) replaced by angular velocity \( \omega = \dot{\theta} \) about the axis of rotation, and \( x \) replaced by angular displacement \( \theta \) (measured with respect to a fixed reference such as the horizontal) about the axis of rotation.

The following exercise considers derivatives of the delta distribution or as is sometimes called, higher order delta distributions.

Exercise 2.2.12 (Higher Order Deltas)

Show that

\[
\delta^{(n)}(t) \overset{\mathcal{L}}{\longrightarrow} s^n \tag{2.62}
\]

This exercise further illustrates the basic idea that differentiation in the t-domain corresponds to multiplication by \( s \) in the s-domain - modulo the initial condition terms elaborated upon within Example 2.2.10.

Comment 2.2.11 (Higher Order Deltas)

Just as \( \delta \) "functions" should be viewed as models for tall, thin, pulse-like signals, derivatives of deltas - called higher order deltas - should be viewed as models for the derivative of tall, thin, pulse-like signals.

Time delays occur in many dynamical systems and processes. They very apparently arise in situations where information is being transmitted across long distances (e.g. Earth to Mars). They are also often used to model chemical processes. The following example presents a fundamental time delay result.

Example 2.2.12 (Time Delay Property)

Show that

\[
f(t - \Delta)1(t - \Delta) \overset{\mathcal{L}}{\longrightarrow} e^{-s\Delta}F(s) \tag{2.63}
\]

for any \( \Delta \geq 0 \).

Solution: Defining \( g(t) \define f(t - \Delta)1(t - \Delta) \) yields

\[
G(s) = (\mathcal{L}g)(s) \define \int_0^\infty f(\tau - \Delta)1(\tau - \Delta)e^{-st}d\tau. \tag{2.64}
\]

Defining \( u \define \tau - \Delta \), yields \( \tau = u + \Delta, d\tau = du \), and

\[
G(s) = (\mathcal{L}g)(s) = \int_{-\Delta}^\infty f(u)1(u)e^{-s(u+\Delta)}du = e^{-s\Delta}\int_{0-}^\infty f(u)e^{-su}du = e^{-s\Delta}F(s), \tag{2.65}
\]

which shows the desired result.
**Comment 2.2.12 (Time Delays and the Delay Operator)**

The above example shows that shifting a signal by an amount \( \Delta \geq 0 \) in the \( t \)-domain corresponds to multiplication by \( e^{-s\Delta} \) in the \( s \)-domain. Given this, \( e^{-s\Delta} \) is sometimes referred to as a delay operator or transfer function.

**Exercise 2.2.13 (Application of Time Delay Property)**

(a) Use the time delay property of the Laplace transform to determine the Laplace transform of the following function \( f(t) = (t-3)1(t-1) \). Answer: \( f(t) = (t-1-2)1(t-1) = (t-1)1(t-1) - 2 \, 1(t-1) \), \( F(s) = e^{-s\frac{1}{2}} - 2e^{-s\frac{1}{2}} \).

(b) Infinite Train of Pulse-Like Signals. Consider the periodic signal \( h(t) = \sum_{n=0}^{\infty} f(t-nT) \) where \( f \) represents a waveform defined on \([0, T]\) and zero elsewhere. Show that \( H(s) = \sum_{n=0}^{\infty} e^{-snT} F(s) = \left[ \frac{1}{1-e^{-sT}} \right] F(s) \).

(c) Show that \( H \) has poles on the imaginary axis at \( s = \pm j\omega_o \) (\( n = 0, 1, \ldots \)) where \( \omega_o = \frac{2\pi}{T} \) is the fundamental frequency of \( h \).

(c) Square Wave. Suppose that \( h(t) = \sum_{n=0}^{\infty} f(t-nT) \) where \( f(t) = 1(t-1-\Delta) \) is a pulse of height 1 and width \( \Delta \) and \( 0 < \Delta < T \). Sketch \( h \). Show that \( F(s) = \left[ \frac{1-e^{-s\Delta}}{s} \right] \) and \( H(s) = \left[ \frac{1-e^{-s\Delta}}{1-e^{-sT}} \right] \left[ \frac{1-e^{-s\Delta}}{s} \right] \).

(c) Half-Wave Rectified Sinusoid. Suppose that \( h(t) = \sum_{n=0}^{\infty} f(t-nT) \) where \( f(t) = \sin(\omega_o t) \left[ 1(t-1-\frac{T}{2}) \right] \) \((\omega_o T = 2\pi)\) represents half a period of \( \sin(\omega_o t) \) on \([0, \frac{T}{2}]\). Sketch \( h \). Show that \( F(s) = \left[ \frac{\omega_o}{\omega_o^2 + s^2} \right] \left[ \frac{1-e^{-s\frac{T}{2}}}{s} \right] \) and \( H(s) = \left[ \frac{1}{1-e^{-s\frac{T}{2}}} \right] \left[ \frac{1-e^{-s\frac{T}{2}}}{s} \right] \).

**Exercise 2.2.14 (Real-Rational Time Delay Approximation)**

In many applications, the irrational function \( e^{-s\Delta} \) is difficult to work with. Often, it is convenient to use the real-rational approximation \( e^{-s\Delta} = \frac{1-e^{-s\Delta}}{s} \approx F_{\Delta}(s) \) \( \overset{\text{def}}{=} \frac{1-s}{1+s^2} = \frac{2}{\omega_o^2 + s^2} \). (a) Show that \( f_{\Delta}(t) = -\delta(t) + \frac{1}{2} e^{-\frac{t}{2}} 1(t) \). (b) Show that \( \int_0^\infty f(t) \, dt = 1 \). (c) Show that \( |F_{\Delta}(j\omega)| = 1 \), just like \( e^{-j\omega\Delta} = 1 \). (d) Show that \( \mathcal{L} \left[ F_{\Delta}(j\omega) \right] = -2 \tan^{-1} \left( \frac{\omega}{\omega_o} \right) \) and that this approximates the phase of \( e^{-j\omega\Delta} \) when \( \omega\Delta \) is small.

Hint: \( \tan^{-1} x \approx x \) for small \( x \).

The following example addresses the mathematical operation known as *convolution*. Convolution is particularly important in the study of linear systems. This is because the output of a system described by a linear ordinary differential equation with constant coefficients and zero initial conditions is given by a convolution integral similar to that addressed in the following example.

**Example 2.2.13 (Time Convolution and Frequency Multiplication Property)**

Show that

\[
\int_0^t f(t-\tau)g(\tau)d\tau \overset{\mathcal{L}}{\leftrightarrow} F(s)G(s).
\]

**Solution:** Let \( h(t) = \int_0^t f(t-v)g(v)dv \) and let \( H(s) \) denote its transform. It then follows that

\[
H(s) = \int_0^\infty h(\tau)e^{-s\tau}d\tau = \int_0^\infty \left[ \int_0^{\tau} f(\tau-v)g(v)dv \right] e^{-s\tau}d\tau = \int_0^\infty \left[ \int_0^{\tau} f(\tau-v)g(v) e^{-s\tau}dv \right] \, dr.
\]

In the above integral, we are vertically summing horizontal differential rectangular elements in the \( v - \tau \) plane that extend from \( v = 0 \) to \( v = \tau \) and have height \( dv \). Switching the order of integration implies that
we are horizontally summing vertical differential rectangular elements in the $v - \tau$ plane that extend from $\tau = v$ to $\infty$ and have width $d\tau$. (Sketch the region in the $v - \tau$ plane.) Noting this yields the following:

$$H(s) = \int_{0}^{\infty} \left[ \int_{v}^{\infty} f(\tau - v) g(v) e^{-s\tau} d\tau \right] dv = \int_{0}^{\infty} \left[ \int_{v}^{\infty} f(\tau - v) e^{-s\tau} d\tau \right] g(v) dv.$$  \hspace{1cm} (2.68)

Letting $u = \tau - v$ and $du = d\tau$ then yields

$$H(s) = \int_{0}^{\infty} \left[ \int_{0}^{\infty} f(u) e^{-s(u+v)} du \right] g(v) dv = \int_{0}^{\infty} \left[ \int_{0}^{\infty} f(u) e^{-su} du \right] e^{-sv} g(v) dv$$

$$= \left[ \int_{0}^{\infty} f(u) e^{-su} du \right] \left[ \int_{0}^{\infty} g(v)e^{-sv} dv \right] = F(s)G(s)$$  \hspace{1cm} (2.70)

which is the desired result.

\[\]  \hspace{1cm} \[\]

**Comment 2.2.13 (A Systems Perspective to Convolution)**

It is well known that continuous-time LTI systems are described in the time domain by an input-output convolution relationship (assuming zero initial conditions) such as that given in Equation 2.66. More precisely, if an input $u(\cdot)$ is applied at $t = 0^-$ to a causal\(^3\) LTI system with impulse response $h(\cdot)$ [214], then the output $y(\cdot)$ of the system (assuming zero initial conditions) is given by

$$y(t) = \int_{0^-}^{t} h(t - \tau) u(\tau) d\tau.$$  \hspace{1cm} (2.71)

From the sifting property of the delta distribution, it follows that when the input is an impulse $u(t) = \delta$, the output is $y(t) = h(t)$ - the impulse response of the system. The above convolution in the t-domain is transformed into a multiplication in the s-domain as follows

$$Y(s) = H(s)U(s).$$  \hspace{1cm} (2.72)

Sometimes it may be quite cumbersome to compute the output $y(\cdot)$ using the convolution integral directly. If $u(\cdot)$ and $h(\cdot)$ are signals with known Laplace transforms, it may be much easier to form $Y(s) = H(s)U(s)$ first and then determine $y(\cdot)$ by using inverse transform methods and a Laplace transform table. Such methods are presented in Section 2.3 (page 67).

**General LTI System.** For a general continuous-time LTI system $H$ with impulse response $h$, we have [214]

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau) u(\tau) d\tau.$$  \hspace{1cm} (2.73)

The system is causal if and only if $h(\cdot)$ is zero for $t < 0$. In such a case, the above becomes

$$y(t) = \int_{-\infty}^{t} h(t - \tau) u(\tau) d\tau.$$  \hspace{1cm} (2.74)

If the system is causal and $u$ is applied on $t \geq 0^-$, then the lower limit becomes $0^-$ and we have

$$y(t) = \int_{0^-}^{t} h(t - \tau) u(\tau) d\tau.$$  \hspace{1cm} (2.75)

---

\(^3\)A causal system is one which produces no output until an input is applied. Consider the LTI system defined by the differential equation $\dot{y} + y = \delta$. Viewing it as a causal system yields the right-sided impulse response $h(t) = e^{-t} 1(t)$. Viewing it as a non-causal or anti-causal system yields the left-sided impulse response $h(t) = -e^{-t} 1(-t)$. Check this by differentiation.
Chapter 2: An Introduction To Laplace Transforms

Exercise 2.2.15 (Application of Convolution Property)
Use the convolution property of the Laplace transform to determine the following function

\[ y(t) = \int_{0^-}^{t} h(t - \tau)u(\tau)d\tau. \]  (2.76)

where \( h(t) = e^{-t}1(t) \) and \( u(t) = 1(t) \). Answer: \( Y(s) = H(s)U(s) = \left[ \frac{1}{s+1} \right] \left[ \frac{1}{s} \right] = \frac{1}{s} - \frac{1}{s+1}, \) \( y(t) = (1 - e^{-t})1(t) \).

Exercise 2.2.16 (Bound on Output of LTI System)
Consider a causal LTI system with zero initial condition and output given by the convolution integral \( y(t) = \int_{0^-}^{t} h(t - \tau)u(\tau)d\tau \).

(a) Show that \( y(t) = \int_{0^-}^{t} h(t - \tau)u(\tau)d\tau = y(t) = \int_{0^-}^{t} h(v)u(t - v)dv. \)

(b) Show that \( |y(t)| \leq \int_{0^-}^{t} |h(v)||u(t - v)| dv. \) Let \( \|y\|_{L^\infty} \overset{\text{def}}{=} \max_{t \geq 0} |y(t)| \) and \( \|h\|_{L^1} \overset{\text{def}}{=} \int_{0^-}^{t} |h(v)| dv. \) Show that

\[ \|y\|_{L^\infty} \leq \|h\|_{L^1} \|u\|_{L^\infty}. \]  (2.77)

This inequality permits us to bound the output given a bound \( \|u\|_{L^\infty} \) on the input; i.e. if \( \|u\|_{L^\infty} \leq B \), then \( \|y\|_{L^\infty} \leq \|h\|_{L^1}, \|u\|_{L^\infty} \leq \|h\|_{L^1} \). B.

(c) Consider the LTI system \( H(s) = \frac{1}{s+1} \). Determine \( h \) and \( \|h\|_{L^1} \). Assuming zero initial condition, determine a bound on the output when \( |u(t)| \leq 10 \) for all \( t \). Answers: \( h = e^{-t}, \|h\|_{L^1} = 1, \|y\|_{L^\infty} \leq 10. \)

Example 2.2.14 (Integration Property)
Show that

\[ \int_{0^-}^{t} f(\tau)d\tau \xleftrightarrow{L} \frac{F(s)}{s}. \]  (2.78)

Solution: This result follows immediately from application of the time convolution frequency multiplication property given in Equation 2.66. The property implies that if a function \( f(\cdot) \) is applied to an integrator at \( t = 0^- \), then the output \( y(\cdot) \) is given by

\[ y(t) = \int_{0^-}^{t} 1(t - \tau)f(\tau)d\tau = \int_{0^-}^{t} f(\tau)d\tau. \]  (2.79)

In the s-domain this becomes

\[ Y(s) = \frac{1}{s}F(s). \]  (2.80)

which is the desired result.

Comment 2.2.14 (\( \frac{1}{s} \) and Integration)
The above example shows that integration in the time domain corresponds to multiplication by \( \frac{1}{s} \) in the s-domain. Given this, \( \frac{1}{s} \) is often referred to as an integral operator. \( \frac{1}{s} \) is henceforth associated with integration in the t-domain.
Exercise 2.2.17 (Application of Integration Property)

Use the integration property of the Laplace transform to determine the Laplace transform of the following function

\[ y(t) = \int_{0}^{t} f(\tau) d\tau. \]  

(2.81)

where \( f(t) = \cos \omega_o t \). Answer: \( Y(s) = F(s) = \frac{1}{s^2 + \omega_o^2} \), \( y(t) = \frac{1}{\omega_o} \sin \omega_o t \).

Theorem 2.2.1 (Output of a Causal LTI System)

Consider a causal LTI system with transfer function \( H \). Let \( h \) denote its impulse response, \( s \) its step response, and \( y \) its general output response.

- **Impulse Response.** The impulse response of the system is given by
  \[ h(t) = \mathcal{L}^{-1}(H(s)) \]  
  or
  \[ h(t) = \frac{d}{dt} s(t). \]  

(2.82)

(2.83)

- **Step Response.** The step response of the system is given by
  \[ s(t) = \mathcal{L}^{-1}\left(\frac{H(s)}{s}\right) \]  
  or
  \[ s(t) = \int_{0}^{t} h(\tau) \, d\tau. \]  

(2.84)

(2.85)

- **General Output Response.** The response \( y \) of the system to an input \( u \) applied on \( t \geq 0 \) is given by
  \[ y(t) = \mathcal{L}^{-1}\left( H(s) \, U(s) \right) \]  
  or
  \[ y(t) = \int_{0}^{t} h(t-\tau) \, u(\tau) \, d\tau. \]  

(2.86)

(2.87)

**Proof:** The first result follows from the convolution property

\[ y(t) = \int_{0}^{t} h(t-\tau) \, u(\tau) \, d\tau \leftrightarrow \mathcal{L} H(s) \, U(s) \]  

(2.88)

with \( u(t) = \delta(t) \) and \( U(s) = 1 \). To prove the next result, consider the step response

\[ s(t) = \int_{0}^{t} h(t-\tau) \, 1(\tau) \, d\tau = \int_{0}^{t} h(t-\tau) \, 1(t-\tau) \, d\tau = \int_{0}^{t} h(\tau) \, d\tau. \]  

(2.89)

The result then follows from the Fundamental Theorem of Calculus. The step response results follow similarly. The general output response results follow directly from the convolution property.
Example 2.2.15 (Initial Value Theorem)

Typically, Equation 2.86 (i.e. \( y(t) = \mathcal{L}^{-1}(H(s) U(s)) \)) is the most useful of the above relationships for determining the output of a causal LTI system. This will be illustrated via many examples.

Exercise 2.2.18 (Translational Motion of a Car)

Consider a car that is initially at rest. Suppose that a constant force is applied to the car at \( t = 0 \). From experience, we expect the car’s velocity to increase until the aerodynamic drag force balances the applied force. Suppose we approximate the acceleration as:

\[
a = ge^{-pt}1(t) \quad \text{where} \quad g, p > 0 \text{ are coefficients that depend on the car’s mass and drag as well as the applied force.} \quad \text{(Note: This is only an approximation. The actual car dynamics are nonlinear. See Exercise 2.17 for additional details.)}
\]

\( (a) \) Determine the velocity \( V \) and displacement \( X \) for \( t \geq 0 \) (i.e. a single integrator since velocity is the integral of acceleration), while \( X \) has a double pole at \( s = -p \) associated with aerodynamic drag. Also note that \( V \) has a single pole at \( s = 0 \) (i.e. a double integrator since displacement is the double integral of acceleration). Do these make sense? Poles always have some physical significance. You should always try to assign physical significance to poles.

The following example shows how one can determine the initial value of a function and its initial derivatives from the function’s s-domain transform.

Example 2.2.15 (Initial Value Theorem)

Suppose that \( F(\cdot) \) is a real-rational function which is at least proper; i.e. the degree of the denominator is greater than or equal to that of the numerator.

\( (a) \) Show that

\[
F(s) = \int_{0^-}^{0^+} f(\tau) \, d\tau + \frac{f(0^+)}{s} + \frac{f'(0^+)}{s^2} + \frac{f''(0^+)}{s^3} + \cdots. \tag{2.90}
\]

Hint: \( F(s) = \int_{0^-}^{0^+} f(\tau) \, d\tau + \int_{0^+}^{\infty} f(\tau) \, e^{-st} \, d\tau, \quad f(t) = f(0^+) + f'(0^+)t + f''(0^+)\frac{t^2}{2} + \cdots \) for \( t > 0 \).

\( (b) \) Show that

\[
\int_{0^-}^{0^+} f(\tau) \, d\tau = F(\infty). \tag{2.91}
\]

The properness assumption guarantees that \( F(\infty) \) is a finite real number. Two cases arise.

- Strictly Proper \( F \): No Impulse at Origin. If the degree of the denominator of \( F \) is greater than that of its numerator, then \( F \) is said to be strictly proper and \( F(\infty) = 0 \). In such a case, \( f \) contains no impulse at the origin.

- Proper \( F \): Impulse at Origin. If the degree of denominator of \( F \) is equal to that of its numerator, then \( F \) is said to be proper and \( F(\infty) \neq 0 \) is a finite real number. In such a case, \( f \) contains an impulse at the origin of size \( F(\infty) \).

\( (c) \) Show that

\[
f(0^+) = \lim_{s \to \infty} s \quad [ \quad F(s) - F(\infty) \quad ]. \tag{2.92}
\]

This result is known as the initial value theorem. Hint: \( F(s) = F(\infty) + \int_{0^+}^{(0^+)} + \frac{f'(0^+)}{s^2} + \frac{f''(0^+)}{s^3} + \cdots \).

\( (d) \) Show that

\[
\dot{f}(0^+) = \lim_{s \to \infty} s^2 \quad [ \quad F(s) - F(\infty) \quad ] - sf(0^+). \tag{2.93}
\]
Section 2.2: Elementary Unilateral Laplace Transforms

(e) Show that
\[ \dot{f}(0^+) = \lim_{s \to \infty} s^3 \left[ F(s) - F(\infty) \right] - s^2 f(0^+) - s \dot{f}(0^+). \] (2.94)

(f) Show that higher order derivatives may be similarly obtained. 

Solution: (a) Since \( F \) is real-rational, \( f \) is continuous on \( t \in (0, \infty) \) and therefore it has a well defined Taylor series expansion about \( t = 0^+ \)

\[ f(t) = f(0^+) + \dot{f}(0^+) t + \frac{\ddot{f}(0^+)}{2!} t^2 + \cdots. \] (2.95)

From this, it follows that

\[ \int_0^\infty f(\tau) e^{-s\tau} d\tau = \frac{f(0^+)}{s} + \frac{\dot{f}(0^+)}{s^2} + \frac{\ddot{f}(0^+)}{s^3} + \cdots. \] (2.96)

This integral result will be used below. Next, we note that the Laplace transform of an arbitrary function \( f \) may be written as follows

\[ F(s) = \int_0^\infty f(\tau) e^{-s\tau} d\tau = \int_0^{0^+} f(\tau) e^{-s\tau} d\tau + \int_{0^+}^{\infty} f(\tau) e^{-s\tau} d\tau \] (2.97)

\[ = \int_0^{0^+} f(\tau) d\tau + \int_{0^+}^{\infty} f(\tau) e^{-s\tau} d\tau. \] (2.98)

Using the integral result from above then yields

\[ F(s) = \int_0^{0^+} f(\tau) d\tau + \frac{f(0^+)}{s} + \frac{\dot{f}(0^+)}{s^2} + \frac{\ddot{f}(0^+)}{s^3} + \cdots. \] (2.99)

This establishes the result in (a).

(b) Letting \( s \to \infty \) in the above yields the result in (b).

(c) Consider the result in (a). Multiplying both sides by \( s \) and using the result from (b) yields

\[ s \left[ F(s) - F(\infty) \right] = f(0^+) + \frac{\dot{f}(0^+)}{s} + \frac{\ddot{f}(0^+)}{s^2} + \cdots. \] (2.100)

Letting \( s \) approach \( \infty \) yields the desired result in (c). The subsequent results follow readily from above. ■

The initial value theorem (proved in the above example) suggests that

the initial behavior of a function \( f \) and its derivatives is related to the behavior of its transform \( F(s) \) for large values of \( s \); i.e. the “high frequency” behavior of \( F \).

Example 2.2.16 (Application of Initial Value Theorem)

Consider the function

\[ f(t) = \delta(t) - e^{-t} 1(t). \] (2.101)

Clearly, \( f(0^+) = -1 \). To obtain this result from knowledge of its transform, we proceed as follows:

\[ F(s) = 1 - \frac{1}{s+1} = \frac{s}{s+1}. \] (2.102)

Given this, it follows from Example 2.2.15 that

\[ f(0^+) = \lim_{s \to \infty} s \left[ F(s) - F(\infty) \right] = \lim_{s \to \infty} s \left[ F(s) - 1 \right] = \lim_{s \to \infty} s \left[ 1 - \frac{1}{s+1} - 1 \right] = -1. \] (2.103)
Chapter 2: An Introduction To Laplace Transforms

The following initial value theorem follows from the ideas presented within Example 2.2.15.

**Theorem 2.2.2 (Initial Value Theorem)**

Suppose that $F$ is the real-rational transform of $f$. It then follows that

$$
\frac{d^N f(0^+)}{dt} = \lim_{s \to \infty} s^{N+1} \left[ F(s) - F(\infty) \right] - s^N f(0^+) - \cdots - s \frac{d^{N-1} f(0^+)}{dt}. \tag{2.104}
$$

Suppose that $F$ has an $m^{th}$ order numerator and an $n^{th}$ order denominator.

- **Proper.** If $F$ is proper ($n = m$, no-pole roll-off), then the $0^{th}$ derivative $f$ has an impulse at $t = 0$. Moreover, the impulse has size (area) $F(\infty)$. Also, $\int_{0^-}^t f(\tau) \, d\tau$ has a discontinuity of size $F(\infty)$ at $t = 0$.

- **1-Pole Roll-Off.** If $F$ has one-pole roll-off ($n = m + 1$), then $F(\infty) = 0$, and $f$ has no impulse at the origin. Moreover, $f$ (0\textsuperscript{th} derivative) is discontinuous at $t = 0$ and

$$
f(0^+) = \lim_{s \to \infty} sF(s). \tag{2.105}
$$

Also, the first derivative $\dot{f}$ has an impulse at $t = 0$.

- **2-Pole Roll-Off.** If $F$ has two-pole roll-off ($n = m + 2$), then $f(0^+) = 0$ and $f$ is continuous at $t = 0$. Moreover, $\dot{f}$ is discontinuous at $t = 0$ and

$$
\dot{f}(0^+) = \lim_{s \to \infty} s^2 F(s). \tag{2.106}
$$

Also, the second derivative $\ddot{f}$ has an impulse at $t = 0$.

- **k-Pole Roll-Off.** More generally, if $F$ has $k$-pole roll-off ($n = m + k$), then $f(0^+) = 0$ and all initial derivatives up to order $k - 2$ are zero and continuous at $t = 0$. Moreover, the $k - 1^{st}$ derivative of $f$ is discontinuous at $t = 0$ and

$$
f^{k-1}(0^+) = \lim_{s \to \infty} s^k F(s). \tag{2.107}
$$

Also, the $k^{th}$ derivative $f^k$ has an impulse at $t = 0$.

From the above, it follows that if we would like $N$ initial derivatives of $f$ to be zero then we require $F$ to have $(N + 2)$-pole roll-off ($n = m + N + 2$).

**Exercise 2.2.19 (Application of Initial Value Theorem)**

Consider $F(s) = \frac{1}{s^4}$. (a) Determine the initial value of $f$ and of its first three derivatives at $t = 0^+$ from $F$. (b) Now determine $f$ and its derivatives directly via inverse transforms. Corroborate the results in (a) by computing the required initial values. (c) Suppose that a signal $f$ represents the displacement of a vehicle. Suppose that we would like the initial displacement, velocity, and acceleration to be continuous and zero at $t = 0$. Show that this will be the case provided that the transform $F$ has at least 4-pole roll-off. (d) Consider $f(t) = \sin \omega_0 t$. Use initial value theorem ideas from Example 2.2.15 to show that $f(0^+) = 0$, $f'(0^+) = \omega_0$, $f''(0^+) = 0$, $f'''(0^+) = -\omega_0^3$. Validate these via direct differentiation as well.

It is now shown how one can determine the final value of a function from its s-domain transform.
Example 2.2.17 (Final Value Theorem)
Suppose that $F(\cdot)$ is a real-rational function corresponding to a causal signal $f$. Show that if the limit

$$f(\infty) \overset{\text{def}}{=} \lim_{t \to \infty} f(t)$$

exists (i.e. is a finite quantity), then

$$f(\infty) = \lim_{s \to 0} sF(s).$$

This result is known as the final value theorem.

Solution: From the derivative theorem, it follows that

$$(L \dot{f})(s) = \int_0^\infty \dot{f}(\tau)e^{-s\tau}d\tau = sF(s) - f(0^-).$$

Letting $s$ approach 0 on each side yields

$$f(\infty) - f(0^-) = \lim_{s \to 0} sF(s) - f(0^-)$$

from which the result follows.

The above example suggests that the behavior of a function $f(t)$ for large $t$ is determined by the behavior of its transform $F(s)$ near $s = 0$; i.e. the “low frequency” behavior of $F$.

Comment 2.2.16 (Application of Final Value Theorem)
Be careful when applying the final value theorem. If $f(t) = e^t1(t)$, then $f(\infty) = \infty$. A “blind” application of the final value theorem, however, yields

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s \left[ \frac{1}{s - 1} \right] = 0.$$ 

This, of course, is ridiculous! Before applying the final value theorem, one must first know that the final value (limit) exists. Always remember that formulae have limitations.

Exercise 2.2.20 (Final Derivative Values)
Suppose that $F(\cdot)$ is a real-rational function corresponding to a causal signal $f$. Let $N = 0, 1, \ldots$. Show that

$$\frac{d^N f(\infty)}{dt} = \lim_{s \to 0} s^{N+1}F(s)$$

provided that the time limit on the left exists.

Comment 2.2.17 (Initial and Final Value Theorems)
Note the similarities between the initial and final value theorems.

- The initial behavior of a signal $f$ depends on the high frequency behavior of $F$. More precisely, the initial value of $\frac{d^N f}{dt}$ depends on the high frequency behavior of $s^{N+1}F$.

- Similarly, the steady state behavior of a signal $f$ depends on the low frequency behavior of $F$. More precisely, the steady state value of $\frac{d^N f}{dt}$ depends on the low frequency behavior of $s^{N+1}F$.
Exercise 2.2.21 (RHP Zero: Non-Minimum Phase Response to Step Input)

Consider the dynamical system \( H(s) = \frac{z}{p + \frac{z}{s}} \) where \( p, z > 0 \). Note that this system possesses a right half plane zero at \( s = z \). Such a system is said to be a non-minimum phase system. Consider its step response \( y \).

(a) Use the final value theorem to show that \( y_{ss} = y(\infty) = 1 \).

(b) Use the initial value theorem to show that \( y(0^+) = -\frac{p}{z} \). This shows that initially, the response moves negatively - opposite where it will be in the steady state. This “inverse behavior,” as it is sometimes referred to, gets worse as the RHP zero gets closer to the origin. The elevator-altitude transfer function for most aircraft has a right half plane zero - implying that when the elevator is deflected downward, the aircraft loses altitude before rising. Exercise ?? (page ??) examines a simple aircraft model.

(c) Show that \( y(t) = 1 - (1 + \frac{p}{z}) e^{-pt} \). Note that \( y_{ss} = y(\infty) = 1 \) and \( y(0^+) = -\frac{p}{z} \) as shown above in (a), (b). ■

2.3 Inverse Laplace Transforms for Real-Rational Functions

The most common method for taking inverse Laplace transforms involves the use of tables. Since real-rational transforms occur frequently in science and engineering applications, efficient methods for taking inverse transforms of real-rational s-domain functions are essential. Such methods are now presented. The methods rely on partial fraction expansion ideas and complex arithmetic. The following examples illustrate typical cases which occur over a broad range of applications.

The following example considers a real-rational function with distinct (simple) real roots (poles).

Example 2.3.1 (Real-Rational Function with Simple Real Poles)

Find the inverse Laplace transform of the following real-rational function

\[
Y(s) = \frac{1}{s(s + 1)(s + 2)}. \tag{2.114}
\]

Solution We begin by expanding \( Y \) in a partial fraction expansion as follows:

\[
Y(s) = \frac{1}{s(s + 1)(s + 2)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2} \tag{2.115}
\]

At this point many authors might find the coefficients \( A, B, C \). Please do not do this! This should be done last!! The next step should be to find the inverse transform of each term - using our knowledge of elementary transform pairs. Specifically, since \( e^{at}1(t) \leftrightarrow \frac{1}{s-a} \), it follows that

\[
y(t) = [A + B e^{-at} + C e^{-2t}]1(t). \tag{2.116}
\]

At this point, we are ready to determine the coefficients \( A, B, C \).

Determining \( A \). To determine \( A \), we consider Equation (2.115). Let’s try to “isolate” \( A \). Toward this end, multiply both sides of Equation (2.115) by \( s \). Doing so, yields

\[
sY(s) = \frac{1}{(s + 1)(s + 2)} = A + \frac{B s}{s + 1} + \frac{C s}{s + 2}. \tag{2.117}
\]

If we now let \( s \rightarrow 0 \), we see that all the terms on the right hand side disappear - except for \( A \); i.e.

\[
\lim_{s \rightarrow 0} sY(s) = \frac{1}{2} = A. \tag{2.118}
\]

We have isolated \( A \)! The above process may be summarized as follows

\[
A = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \left[ \frac{1}{s(s + 1)(s + 2)} \right] = \lim_{s \rightarrow 0} \left( \frac{1}{(s + 1)(s + 2)} \right) = \frac{1}{2}. \tag{2.119}
\]
The coefficients $B$ and $C$ are found in an identical manner. We summarize as follows.

**Determining $B$.** To find $B$, one proceeds as follows

$$ B = \lim_{s \rightarrow -1} (s+1)Y(s) = \lim_{s \rightarrow -1} (s+1) \left[ \frac{1}{s(s+1)(s+2)} \right] = \lim_{s \rightarrow -1} \frac{1}{s(s+2)} = -1. \quad (2.120) $$

**Determining $C$.** To find $C$, one proceeds as follows

$$ C = \lim_{s \rightarrow -2} (s+2)Y(s) = \lim_{s \rightarrow -2} (s+2) \left[ \frac{1}{s(s+1)(s+2)} \right] = \lim_{s \rightarrow -2} \frac{1}{s(s+1)} = \frac{1}{2}. \quad (2.121) $$

Given the above, it follows that

$$ y(t) = \left[ \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \right] 1(t). \quad (2.122) $$

Note: Show that this is also the step response of the LTI system with transfer function $H(s) = \frac{1}{(s+1)(s+2)}$ and associated differential equation $\ddot{y} + 3\dot{y} + 2y = u$.

**Exercise 2.3.1 (First Order Ordinary Differential Equation: Method of Transfer Function)**

**Solve the following ordinary differential equation for $y$**

$$ \dot{y} + 2y = 6 \, u(t) \quad y(0^-) = 0 \quad (2.123) $$

for $t > 0$ with (a) $u(t) = 10 \, 1(t)$, (b) $u(t) = 5 \cos(100t - 30^\circ)1(t)$, (c) $u(t) = 10 \, 1(t) + 5 \cos(100t - 30^\circ)1(t)$. \[\blacksquare\]

**Answers:** Show that assuming zero initial conditions, $Y(s) = \left[ \frac{6}{s+2} \right] U(s)$. Given this, we define

$$ H(s) \overset{\text{def}}{=} \frac{Y(s)}{U(s)} \bigg|_{\text{zero initial conditions}} = \frac{6}{s+2} \quad (2.124) $$

- the so-called transfer function associated with the differential equation. As mentioned earlier, transfer function concepts are very useful when dealing with linear ordinary differential equations with constant coefficients. This exercise is intended to illustrate how this is so. Given the above, we have the following:

(a) $Y(s) = \left[ \frac{6}{s+2} \right] \frac{10}{s} = \frac{A}{s} + \frac{B}{s+2}$. This yields $y(t) = A + Be^{-2t} = 10H(0) + Be^{-2t}$ where $A = 10H(0)$ and $B = (s+2)Y(s)|_{s=-2}$. This shows that if we apply a step of size 10, then in the steady state (after all transients have decayed), we are left with $y_{ss} = 10H(0)$ where $H(0) = \frac{B}{A} = 3$. Verify this - as it is a very important concept. It is a special case of the so-called Method of the Transfer Function (MOTF) to be introduced in Section ???. (Also, see Appendix ???, section ?? on the solution of linear ordinary differential equations with constant coefficients for an introduction to MOTF concepts.) The $Be^{-2t}$ term is a transient term - associated with transfer function’s pole at $s = -2$. Such a term is generally present in $y$ when a forcing function $u$ is applied to the differential equation.

(b) $Y(s) = \left[ \frac{6}{s+2} \right] 5 \left( \frac{\cos(-30^\circ) \, s - 100 \sin(-30^\circ)}{s^2 + 100} \right) = \frac{B}{s+2} + \frac{C}{s-j100} + \ast$. This yields $y(t) = Be^{-2t} + 2|C| \cos(100t + \angle C) = Be^{-2t} + 5|H(j100)| \cos(100t - 30^\circ + \angle H(j100))$ where $B = (s+2)Y(s)|_{s=-2}$ and $H(j100) \approx \frac{6}{100} e^{-j90^\circ}$. (See Appendix ?? for an overview of complex arithmetic; particularly Exercise ???)

The above shows that if we apply a sinusoid $5 \cos(100t - 30^\circ)1(t)$, then in the steady state (after all transients have decayed) we are left with a sinusoid $y_{ss} = 5|H(j100)| \cos(100t - 30^\circ + \angle H(j100))$. Verify this - as it is the key to sinusoidal analysis of LTI systems. This method for determining $y_{ss}$ is referred to as the Method of the Transfer Function (MOTF).
Chapter 2: An Introduction To Laplace Transforms

(c) \( Y(s) = \left[ \frac{6}{s+2} \right] \left[ \frac{10}{s} + 5 \left( \frac{\cos(-30^\circ) s^{-100} \sin(-30^\circ)}{s^2 + 10^4} \right) \right] = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-100} + \ast. \) This yields \( y(t) = A + Be^{-2t} + 2|C| \cos(100t + \angle C) = 10H(0) + Be^{-2t} + 5|H(j100)| \cos(100t - 30^\circ + \angle H(j100)) \approx 10(3) + Be^{-2t} + 5\frac{6}{100} \cos(100t - 30^\circ - 90^\circ) \). Note that the steady state response is given by \( y_{ss} = 10H(0) + 5|H(j100)| \cos(100t - 30^\circ + \angle H(j100)) \approx 10(3) + 5\frac{6}{100} \cos(100t - 30^\circ - 90^\circ) \). This follows from the so-called Method of the Transfer Function (MOTF) - see Section ?? for additional details. (See Appendix ?? for an overview of complex arithmetic; particularly Exercise ??). Also, see Appendix ??, section ?? on the solution of linear ordinary differential equations with constant coefficients for an introduction to MOTF concepts.)

First Order Strictly Proper System. Now consider an LTI system with first order transfer function \( H(s) = \frac{b}{s+a} \) where \( a, b > 0 \). Since \( H(0) = 0 \), we say that \( H \) is strictly proper.

(d) Show that the steady state response to a unit step is given by \( y_{ss} = H(0) = \frac{b}{a} \) - the so-called dc gain of the system. Show that the step response is given by \( y(t) = H(0)(1 - e^{-at}) \).

(e) Show that the steady state response to a sinusoidal input \( u(t) = A \cos(\omega_o t + \theta) \) is given by \( y_{ss} = A|H(j\omega_o)| \cos(\omega_o t + \theta + \angle H(j\omega_o)) \) where \( |H(j\omega_o)| = \frac{b}{\sqrt{\omega_o^2 + a^2}} \) and \( \angle H(j\omega_o) = -\tan^{-1}(\frac{a}{b}) \). The function \( H(j\omega_o) \) is called the frequency response of the system. \( |H(j\omega_o)| \) is called the magnitude response. \( \angle H(j\omega_o) \) is called the phase response. Now show that the response to the above sinusoid is given by \( y(t) = \left( \frac{-Ah(a \cos \theta + \omega_o \sin \theta)}{\omega_o^2 + a^2} \right) e^{-at} + A|H(j\omega_o)| \cos(\omega_o t + \theta + \angle H(j\omega_o)) \).

The above shows that the MOTF can be very useful for computing the steady state response of a stable system driven by steps and sinusoids. MOTF is one of the most important concepts in the analysis of LTI systems - see Section ?? for additional details. (Also, see Appendix ??, section ?? on the solution of linear ordinary differential equations with constant coefficients for an introduction to MOTF concepts.)

Exercise 2.3.2 (Second Order Ordinary Differential Equation: Real Roots)

Solve the following ordinary differential equation for \( y \)

\[
\ddot{y} + 3\dot{y} + 2y = 21(t) \quad y(0^-) = y_0 \quad \dot{y}(0^-) = \dot{y}_0 \quad (2.125)
\]

for \( t > 0 \). Viewing \( u(t) = 1(t) \) as the input, what is the associated transfer function \( H \)? Show that \( y_{ss} = H(0) \). \( H(0) \) is called the dc gain of the system.

Answer: \( y(t) = 1 + (\dot{y}_0 + 2y_0 - \dot{y}_0 - y_0)e^{-t} + (1 - \dot{y}_0 - y_0)e^{-2t} \). \( H(s) = \frac{2}{s^2 + 3s + 2} = \frac{2}{(s+1)(s+2)} \). When all initial conditions are zero, how does this answer differ from that obtained in Exercise 2.3.1? (Note the factor 2 in front of \( 1(t) \) in differential equation.) What is the step response of the system with transfer function \( H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} \) and dc gain \( H(0) = \frac{1}{2} \)?

The following example considers a real-rational function with repeated real roots (poles).

Example 2.3.2 (Real-Rational Function with Repeated Real Poles)

Find the inverse Laplace transform of the following real-rational function

\[
Y(s) = \frac{1}{s(s+1)(s+2)^2}. \quad (2.126)
\]

Solution We begin by expanding \( Y \) in a partial fraction expansion as follows:

\[
Y(s) = \frac{1}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)} + \frac{E}{(s+2)^2} + \frac{F}{s+2}. \quad (2.127)
\]
At this point, we do not compute the coefficients! Instead, we recall the elementary transform pair
\[ \frac{t^n}{n!}e^{at}1(t) \leftrightarrow \frac{1}{(s-a)^{n+1}}. \] (2.128)

Given this, it follows that
\[ y(t) = \left[ A + Be^{-t} + \frac{C}{3!}e^{-2t} + \frac{D}{2!}e^{-2t} + \frac{E}{1!}e^{-2t} + Fe^{-2t} \right]1(t). \] (2.129)

The coefficients \( A \) and \( B \) are easily found as follows:
\[ A = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \left[ \frac{1}{s(s+1)(s+2)^2} \right] = \lim_{s \to 0} \frac{1}{s+1} = \frac{1}{16} \] (2.130)
\[ B = \lim_{s \to -1} (s+1)Y(s) = \lim_{s \to -1} (s+1) \left[ \frac{1}{s(s+1)(s+2)^2} \right] = \lim_{s \to -1} \frac{1}{s(s+2)^2} = -1 \] (2.131)

Now let’s find \( C, D, E, F \).

**Determining C.** To determine \( C \), we consider Equation (2.127). Let’s try to “isolate” \( C \). Toward this end, multiply both sides of Equation (2.127) by \((s+2)^4\). Doing so yields
\[ (s+2)^4Y(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + Ds + E + Fs + G. \] (2.132)

Letting \( s \to -2 \) causes all of the terms to disappear – except for \( C \). The process is summarized as follows:
\[ C = \lim_{s \to -2} (s+2)^4Y(s) = \lim_{s \to -2} (s+2)^4 \left[ \frac{1}{s(s+1)(s+2)^2} \right] = \lim_{s \to -2} \frac{1}{s+1} = \frac{1}{2}. \] (2.133)

**Determining D.** To determine \( D \), we begin with
\[ (s+2)^4Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + Ds + E + Fs + G. \] (2.134)

To “isolate” \( D \), we differentiate both sides with respect to \( s \). Doing so yields
\[ \frac{d}{ds} [(s+2)^4Y(s)] = \frac{d}{ds} \left[ \frac{A}{s} + \frac{B}{s+1} \right] + D + 2E(s+2) + 3F(s+2)^2. \] (2.135)

Letting \( s \to -2 \) now causes all of the terms to disappear – except for \( D \). The process is summarized as follows:
\[ D = \lim_{s \to -2} \frac{d}{ds} [(s+2)^4Y(s)] = \lim_{s \to -2} \frac{d}{ds} \left[ \frac{1}{s+2} \right] = \frac{-1}{(4-2)^2} (-4 + 1) = \frac{3}{4}. \] (2.136)

The differentiation performed above was fairly easy to carry out by hand. In general, however, the required differentiation (see differentiation required for coefficients \( E \) and \( F \) below) may be very difficult to carry out by hand. In such a case, a symbolic computing language such as Maple or Mathematica would be very appropriate to carry out the differentiation.

**Determining E.** To determine \( E \), we begin with
\[ \frac{d}{ds} [(s+2)^4Y(s)] = \frac{d}{ds} \left[ \frac{A}{s} + \frac{B}{s+1} \right] + D + 2E(s+2) + 3F(s+2)^2. \] (2.137)

To “isolate” \( E \), we differentiate both sides with respect to \( s \). Doing so yields
\[ \frac{d^2}{ds^2} [(s+2)^4Y(s)] = \frac{d^2}{ds^2} \left[ \frac{A}{s} + \frac{B}{s+1} \right] + 0 + 2E + 3F(s+2)^2. \] (2.138)
Chapter 2: An Introduction To Laplace Transforms

Letting $s \rightarrow -2$ causes all of the terms to disappear - except for $2!E$. One can then find $E$ by solving

$$ E = \frac{1}{2!} \lim_{s \rightarrow -2} \frac{d^2}{ds^2} [(s + 2)^4 Y(s)] $$

(2.139)
on a computer. You really wouldn’t want to do this by hand - would you? In general, the differentiation can be a real mess. We must be very grateful for a nice compact formula which can be easily implemented on a computer!

Determining $F$. To determine $F$, we begin with

$$ \frac{d^2}{ds^2} [(s + 2)^4 Y(s)] = \frac{d^2}{ds^2} \left[ \frac{A}{s} (s + 2)^4 \right] + \frac{d^2}{ds^2} \left[ \frac{B}{s + 1} (s + 2)^4 \right] + 0 + 0 + 2!E + 3!F(s + 2). $$

(2.140)

To “isolate” $F$, we differentiate both sides with respect to $s$. Doing so yields

$$ \frac{d^3}{ds^3} [(s + 2)^4 Y(s)] = \frac{d^3}{ds^3} \left[ \frac{A}{s} (s + 2)^4 \right] + \frac{d^3}{ds^3} \left[ \frac{B}{s + 1} (s + 2)^4 \right] + 0 + 0 + 0 + 3!F. $$

(2.141)

Letting $s \rightarrow -2$ causes all of the terms to disappear - except for $3!F$. One can then find $F$ by solving

$$ F = \frac{1}{3!} \lim_{s \rightarrow -2} \frac{d^3}{ds^3} [(s + 2)^4 Y(s)] $$

(2.142)
on a computer! The following MATLAB command sequence can be used to compute $A, B, C, D, E, F$.

num = 1; % Form numerator polynomial
den = conv([1 0], [1 1]); % Form s(s+1)
roots(den); % Verify Roots
den = conv(den, [1 4 4]); % Form s(s+1)(s+2)^2
roots(den); % Verify Roots
den = conv(den, [1 4 4]); % Form s(s+1)(s+2)^4
roots(den); % Verify Roots
[r p k] = residue(num,den); % Compute Residues r, roots p, and remainder k(s)

Implementing this sequence yields

$$ r = \begin{bmatrix} 0.9375 & 0.8750 & 0.7500 & 0.5000 & -1.0000 & 0.0625 \end{bmatrix}^T $$

(2.143)

$$ p = \begin{bmatrix} -2.0000 & -2.0000 & -2.0000 & -2.0000 & -1.0000 & 0 \end{bmatrix}^T. $$

(2.144)

This, then implies that

$$ A = \frac{1}{16} = 0.0625, \quad B = -1, \quad C = 0.5, \quad D = 0.75, \quad E = 0.875, \quad F = 0.9375. $$

(2.145)

Given this, it follows that

$$ y(t) = \left[ 0.0625 - e^{-t} + 0.5 \frac{t^3}{3!} e^{-2t} + 0.75 \frac{t^2}{2!} e^{-2t} + 0.875 \frac{t}{1!} e^{-2t} + 0.9375 e^{-2t} \right] 1(t). $$

(2.146)

Final Note: Show that $y$ is the step response of the LTI system with transfer function $H(s) = \frac{1}{(s+1)(s+2)^4}$. ■

Exercise 2.3.3 (Ordinary Differential Equation: Repeated Roots)

Solve the following ordinary differential equation for $y$ \begin{equation}
\frac{d^5 y}{dt^5} + a_4 \frac{d^4 y}{dt^4} + a_3 \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + y = 1(t) \end{equation}

for $t > 0$ where $\Phi(s) = s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = (s + 1)(s + 2)^4$ is the associated characteristic polynomial. All initial conditions are assumed to be zero at $t = 0^-$. ■
Section 2.3: Inverse Laplace Transforms for Real-Rational Functions

Hint: See Example 2.3.2

The following example considers a real-rational function with complex conjugate roots (poles) on the imaginary axis.

Example 2.3.3 (Real-Rational Function with Complex Conjugate Imaginary Poles.)

Find the inverse Laplace transform of the following real-rational function

\[ Y(s) = \frac{\omega_o}{s(s^2 + \omega_o^2)}. \]  

Solution Noting that \( s^2 + \omega_o^2 = (s - j\omega_o)(s + j\omega_o) \), we begin by expanding \( Y \) in a partial fraction expansion as follows:

\[ Y(s) = \frac{\omega_o}{s(s^2 + \omega_o^2)} = \frac{A}{s} + \frac{B}{s - j\omega_o} + \frac{C}{s + j\omega_o}. \]  

At this point, we do not determine the coefficients \( A, B, C \) - they are easily computed using standard partial fraction expansion formulae for simple (albeit complex) roots. We do note, however, that \( C \) will be the complex conjugate of \( B \). (This is expected since \( Y \) is real-rational and therefore, we expect the terms in its partial fraction expansion to occur in complex conjugate pairs. How else could we get a real-rational function \( Y? \) This will be verified below. With this key fact in hand, the next step is to take inverse transforms - term-by-term. Doing so yields

\[ y(t) = [A + Be^{j\omega_o t} + \star] 1(t) \]  

where \( \star \) denotes the complex conjugate of the preceding term.

Next, we note that the quantity \( B \) - which, in general, will be complex - can be written as

\[ B = |B|e^{j\angle B}. \]  

Substituting this into the previous equation yields

\[ y(t) = [A + |B|e^{j\angle B} e^{j\omega_o t} + \star] 1(t). \]  

Rewriting this expression so that all of the “\( j \) terms” appear next to one another then yields

\[ y(t) = [A + |B| e^{j(\omega_o t + \angle B)} + \star] 1(t). \]  

Now, we recall Euler’s formula (see Equation (?)): for a complex quantity \( z \), and its conjugate \( \bar{z} \), it follows that:

\[ z + \bar{z} = 2Re z = 2|z| \cos \angle z. \]  

Letting \( z = |B| e^{j(\omega_o t + \angle B)} \), then yields

\[ y(t) = [A + z + \bar{z}] 1(t) = [A + 2|z| \cos \angle z] 1(t) = [A + 2|B| \cos(\omega_o t + \angle B)] 1(t). \]  

The coefficients \( A, B, C \) are found as follows:

\[ A = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \frac{\omega_o}{s(s^2 + \omega_o^2)} = \lim_{s \to 0} \frac{\omega_o}{s^2 + \omega_o^2} = \frac{\omega_o}{\omega_o} = 1 \]  

\[ B = \lim_{s \to j\omega_o} (s - j\omega_o)Y(s) = \lim_{s \to j\omega_o} \frac{\omega_o}{s(s + j\omega_o)} = \frac{1}{(1e^{j90^\circ})(2\omega_o e^{j90^\circ})} = \frac{1}{2\omega_o} e^{-j180^\circ}. \]  

\[ C = \lim_{s \to -j\omega_o} (s + j\omega_o)Y(s) = \lim_{s \to -j\omega_o} \frac{\omega_o}{s(s - j\omega_o)} = \frac{1}{(1e^{-j90^\circ})(2\omega_o e^{-j90^\circ})} = \frac{1}{2\omega_o} e^{j180^\circ} = \bar{B}. \]  

Note that \( C = \bar{B} \) (conjugate of \( B \)), as expected.
Chapter 2: An Introduction To Laplace Transforms

Given the above calculations, it follows that

\[ y(t) = \left[ \frac{1}{\omega_o} + \frac{1}{\omega_o} \cos(\omega_o t - 180^\circ) \right] 1(t) = \left[ \frac{1}{\omega_o} - \frac{1}{\omega_o} \cos \omega_o t \right] 1(t). \quad (2.159) \]

Note that this is what we expect by applying the result from Example 2.2.14 to \( Y(s) = \frac{\omega_o s}{s^2 + \omega_o^2} \). Doing so yields

\[ y(t) = \int_0^t \sin \omega_o t \, dt = \left[ \frac{1}{\omega_o} \cos \omega_o t \right] \bigg|_0^t = \left[ \frac{1}{\omega_o} - \frac{1}{\omega_o} \cos \omega_o t \right] 1(t), \quad (2.160) \]

which agrees with the result obtained above!

Final Note: Show that \( y \) is the step response of the LTI system with transfer function \( H(s) = \frac{\omega_o}{s^2 + \omega_o^2} \) (an oscillator), (2) \( y \) is the response of the LTI system with transfer function \( H(s) = \frac{1}{s} \) (an integrator) to the sinusoidal input \( u = \sin \omega_o t \). Specifically, show that

\[ y(t) = \left[ \frac{1}{\omega_o} + |H(j\omega_o)| \sin (\omega_o t + \angle H(j\omega_o)) \right] 1(t) = \left[ \frac{1}{\omega_o} + \frac{1}{\omega_o} \sin (\omega_o t - 90^\circ) \right] 1(t), \quad (2.161) \]

which agrees with the result obtained above!

The following exercise considers a model for a simple harmonic oscillator. Such models arise in applications involving oscillatory phenomena; e.g. automobile suspension system.

Exercise 2.3.4 (Second Order Ordinary Differential Equation: Imaginary Roots.)
Solve the following ordinary differential equation for \( y \)

\[ \ddot{y} + \omega_o^2 y = \omega_o \, u(t) \quad (2.163) \]

for \( t > 0 \) where \( u(t) = 1(t) \). Assume that all initial conditions are zero at \( t = 0^- \).

Hint: See Example 2.3.3, page 72.

The following example considers a real-rational function with complex conjugate roots (poles).

Example 2.3.4 (Real-Rational Function with Complex Conjugate Poles)
Find the inverse Laplace transform of the following real-rational function

\[ Y(s) = \frac{1}{s(s^2 + s + 1)}. \quad (2.164) \]

Solution

Noting that \( s^2 + s + 1 = (s + \frac{1}{2} - j\frac{\sqrt{3}}{2})(s + \frac{1}{2} + j\frac{\sqrt{3}}{2}) \), we begin by expanding \( Y \) in a partial fraction expansion as follows:

\[ Y(s) = \frac{1}{s(s^2 + s + 1)} = \frac{A}{s} + \frac{B}{s + \frac{1}{2} - j\frac{\sqrt{3}}{2}} + \frac{C}{s + \frac{1}{2} + j\frac{\sqrt{3}}{2}}. \quad (2.165) \]

At this point, we do not determine the coefficients \( A, B, C \) - they are easily computed using standard partial fraction expansion formulae for simple (albeit complex) roots. We do note, however, that \( C \) will be the...
complex conjugate of \( B \). This will be verified below. With this key fact in hand, the next step is to take inverse transforms - term-by-term. Doing so yields

\[
y(t) = \left[ A + B e^{\left(-\frac{1}{2} + j \frac{\sqrt{2}}{2}\right)t} + * \right] 1(t)
\]  

(2.166)

where \( * \) denotes the complex conjugate of the preceding term. Next we note that the quantity \( B \) - which, in general, will be complex - can be written as

\[
B = |B| e^{j \angle B}.
\]  

(2.167)

Substituting this into the previous equation yields

\[
y(t) = \left[ A + |B| e^{j \angle B} e^{\left(-\frac{1}{2} + j \frac{\sqrt{2}}{2}\right)t} + * \right] 1(t).
\]  

(2.168)

Rewriting this expression so that all of the “\( j \) terms” appear next to one another then yields

\[
y(t) = \left[ A + |B| e^{\left(-\frac{1}{2} t + j \angle B \right)} + * \right] 1(t).
\]  

(2.169)

Now, we recall Euler’s formula (see Equation (2.7)). For a complex quantity \( z \), and its conjugate \( \bar{z} \), it follows that:

\[
z + \bar{z} = 2 \text{Re} z = 2|z| \cos \angle z.
\]  

(2.170)

Letting \( z = |B| e^{-\frac{1}{2} t} e^{j (\angle B + \angle \mathcal{L} B)} \), then yields

\[
y(t) = [A + z + \bar{z}] 1(t) = [A + 2|z| \cos \angle z] 1(t) = \left[ A + 2|B| e^{-\frac{1}{2} t} \cos \left(\frac{\sqrt{3}}{2} t + \angle \mathcal{L} B\right)\right] 1(t). \tag{2.171}
\]

The coefficients \( A, B, C \) are found as follows:

\[
A = \lim_{s \to 0} s Y(s) = \lim_{s \to 0} s \left[ \frac{1}{s(s^2 + s + 1)} \right] = \lim_{s \to 0} \frac{1}{s^2 + s + 1} = 1.
\]  

(2.172)

\[
B = \lim_{s \to -\frac{1}{2} + j \frac{\sqrt{2}}{2}} \left( s + \frac{1}{2} - j \frac{\sqrt{3}}{2} \right) Y(s) = \lim_{s \to -\frac{1}{2} + j \frac{\sqrt{2}}{2}} \left( s + \frac{1}{2} - j \frac{\sqrt{3}}{2} \right) = \lim_{s \to 0} \frac{1}{s(s^2 + s + 1)}
\]  

(2.173)

\[
C = \lim_{s \to -\frac{1}{2} - j \frac{\sqrt{2}}{2}} \left( s + \frac{1}{2} + j \frac{\sqrt{3}}{2} \right) Y(s) = \lim_{s \to -\frac{1}{2} - j \frac{\sqrt{2}}{2}} \left( s + \frac{1}{2} + j \frac{\sqrt{3}}{2} \right) = \lim_{s \to 0} \frac{1}{s(s^2 + s + 1)}
\]  

(2.174)

Note that \( C = \bar{B} \) (conjugate of \( B \)), as expected.

Given the above calculations, it follows that

\[
y(t) = \left[ 1 + \frac{2}{\sqrt{3}} e^{-\frac{1}{2} t} \cos \left(\frac{\sqrt{3}}{2} t - 210^\circ\right) \right] 1(t). \tag{2.177}
\]
about \( M_p = 0.163 \) (or 16.3\%) at \( t_p = 3.63 \) seconds. These parameters are explained in Example ??, page ?? which addresses the step response of second order systems in so-called “standard form.” Such systems have the “standard second order form differential equation representation”

\[
\ddot{y} + 2\zeta \omega_n \dot{y} + \omega_n^2 y = \omega_n^2 u
\]  

or equivalently the “standard second order form transfer function representation”

\[
H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

where \( \zeta \) is the damping factor and \( \omega_n \) is the undamped natural frequency. We have \( Y(s) = \frac{1}{s(s^2 + s + 1)} = H(s) \frac{1}{s} \)

It thus follows that \( \omega_n = 1 \) and \( \zeta = 0.5 \).

For such a system (Example ??, page ??), the overshoot to a step input - denoted \( M_p \) - is given by Equation (2.181)

\[
M_p = e^{-\pi \zeta \sqrt{1 - \zeta^2}} \approx 0.163
\]

and the time-to-peak - denoted \( t_p \) - is given by Equation (2.182)

\[
t_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\sqrt{2}} \approx 3.63 \text{ seconds}
\]

See Example ??, page ?? for details. The following MATLAB command sequence was used to generate Figure 2.1.
Exercise 2.3.5 (Second Order Ordinary Differential Equation: Complex Roots.)

Solve the following ordinary differential equation for $y$

\[ \ddot{y} + \dot{y} + y = u(t) \quad y(0^-) = y_o \quad \dot{y}(0^-) = \dot{y}_o \]  

(2.183)

for $t > 0$ where $u(t) = 1(t)$. Here, $y$ might represent the vertical displacement of a vehicle with an under-damped suspension system subject to a unit step disturbance.

Answer: $Y = \frac{s^2 y_o + \dot{y}_o + y_o}{s^2 + 4s + 13} + \left[ \frac{1}{s^2 + 4s + 13} \right] \frac{1}{s} = \frac{s(s y_o + \dot{y}_o + y_o) + 1}{s(s^2 + 4s + 13)} = A + \frac{B}{s + 2 + j \sqrt{2}} + \ast$. 

$y = A + 2|B|e^{-0.5t} \cos \left( \frac{\sqrt{3}}{2} t + B \right), \quad A = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sH(s) \frac{1}{s} = H(0) = 1$ where $H(s) = \frac{1}{s^2 + 4s + 13}$.

$B = \left. \frac{s(s y_o + \dot{y}_o + y_o) + 1}{s(s^2 + 4s + 13)} \right|_{s = -\frac{1}{2} + j \frac{\sqrt{2}}{2}} = \frac{1}{1 + \frac{1}{\sqrt{3} e^{j90^\circ}}} = \frac{\sqrt{3}}{2} \sqrt{3} e^{j90^\circ} = \frac{1}{e^{j90^\circ}}$

and $y = 1 + \frac{2}{\sqrt{3}} e^{-0.5t} \cos \left( \frac{\sqrt{3}}{2} t - 210^\circ \right)$. See Example 2.3.4, page 73.

(b) Repeat the above for $\ddot{y} + 4\dot{y} + 13y = 13 \, 1(t)$  

(2.184)

Answer: $Y = \frac{s^2 y_o + \dot{y}_o + y_o}{s^2 + 4s + 13} + \left[ \frac{\sqrt{3}}{s^2 + 4s + 13} \right] \frac{1}{s} = \frac{s(s y_o + \dot{y}_o + y_o) + 13}{s(s^2 + 4s + 13)} = A + \frac{B}{s + 2 - j \sqrt{2}} + \ast$. 

$y = A + 2|B|e^{-2t} \cos(3t + B), \quad A = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sH(s) \frac{1}{s} = H(0) = 1$ where $H(s) = \frac{13}{s^2 + 4s + 13}$.

$B = \left. \frac{s(s y_o + \dot{y}_o + y_o) + 13}{s(s^2 + 4s + 13)} \right|_{s = -2 + j \sqrt{2}} = \frac{1}{e^{j120^\circ} \sqrt{3} e^{j90^\circ}} = \frac{\sqrt{3}}{2} e^{j120^\circ}$

$\frac{1}{e^{j90^\circ}}$

and $y = 1 + \frac{2}{\sqrt{3}} e^{-2t} \cos \left( \frac{\sqrt{3}}{2} t - 210^\circ \right)$. See Example 2.3.4, page 73.

(c) Repeat the above for $\ddot{y} + 4\dot{y} + 13y = 39 \, 1(t)$  

(2.185)

Answer: $Y = \frac{s^2 y_o + \dot{y}_o + y_o}{s^2 + 4s + 13} + \left[ \frac{\sqrt{3}}{s^2 + 4s + 13} \right] \frac{1}{s} = \frac{s(s y_o + \dot{y}_o + y_o) + 39}{s(s^2 + 4s + 13)} = A + \frac{B}{s + 2 - j \sqrt{2}} + \ast$. 

$y = A + 2|B|e^{-2t} \cos(3t + B), \quad A = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sH(s) \frac{1}{s} = H(0) = 3$ where $H(s) = \frac{39}{s^2 + 4s + 13}$.

$B = \left. \frac{s(s y_o + \dot{y}_o + y_o) + 39}{s(s^2 + 4s + 13)} \right|_{s = -2 + j \sqrt{2}} = \frac{1}{e^{j120^\circ} \sqrt{3} e^{j90^\circ}} = \frac{\sqrt{3}}{2} e^{j120^\circ}$

$\frac{1}{e^{j90^\circ}}$

and $y = 1 + \frac{2}{\sqrt{3}} e^{-2t} \cos \left( \frac{\sqrt{3}}{2} t - 210^\circ \right)$. See Example 2.3.4, page 73.

The following example considers a real-rational function which is improper; i.e. the degree of the numerator is greater than that of the denominator.
Example 2.3.5 (Improper Real-Rational Functions)

Find the inverse Laplace transform of the following real-rational function

(a) \( Y(s) = \frac{10s + 20}{2s + 6} \)

(b) \( Y(s) = \frac{s^2 + s + 1}{s^2 + 2s + 1} \)

Solution

The key point to make here is that: One cannot perform a standard partial fraction expansion unless the degree of the numerator is less than that of the denominator.

When the degree of the numerator is greater than or equal to that of the denominator, one must first perform a long division before initiating a partial fraction expansion.

(a) Since \( Y(s) = \frac{10s + 20}{2s + 6} \) has a numerator with degree equal to that of the denominator, we cannot proceed with a standard partial fraction expansion. We must first perform a long division. The division required here is very easy to carry out as the following steps indicate:

\[
Y(s) = \frac{10s + 20}{2s + 6} = \frac{10}{2} \left( \frac{s + 2}{s + 3} \right) = \frac{5}{s + 3}
\]

From this, it follows that

\[
y(t) = 5\delta(t) - 5e^{-3t}1(t).
\]

(b) Since \( Y(s) = \frac{s^2 + s + 1}{s^2 + 2s + 1} \) has a numerator with degree greater than that of the denominator, we cannot proceed with a standard partial fraction expansion. The denominator must first be divided into the numerator. The result of the long division takes the form

\[
Y(s) = \frac{s^2 + s + 1}{s^2 + 2s + 1} = As + B + \frac{C}{s + 2}
\]

(Note: Getting the form of the division should have higher priority than computing the coefficients.) Given the above form, it follows that

\[
y(t) = A\delta(t) + B\delta(t) + Ce^{-2t}1(t).
\]

Now compute the coefficients \( A, B, C \). They can be found by actually performing the long division or by executing the following MATLAB command sequence

```matlab
num = [1 1 1]; % Form s^2 + s + 1
den = [1 2 ]; % Form s + 2
[r p k] = residue(num,den) % Compute residues, poles, and dividend
```

The result yields a remainder (or residue) \( r = 3 \), an associated pole \( p = -2 \), and a dividend \( k = [1 -1] \) (i.e. \( k(s) = s - 1 \)). This implies that \( A = 1, B = -1, \) and \( C = 3 \). It thus follows that

\[
y(t) = \delta(t) - \delta(t) + 3e^{-2t}1(t).
\]

The following example is loaded with “good stuff” (e.g. repeated real, imaginary, and complex poles).

Example 2.3.6 (Real-Rational Function with Repeated Real, Imaginary, and Complex Poles)

Find the form of the inverse Laplace transform of the following real-rational function

\[
Y(s) = \frac{1}{s^2(s + 1)(s^2 + 100)^2(s^2 + s + 1)^3}
\]

Determine all of the “important” coefficients using MATLAB. What issues arise in determining the other coefficients?
Solution: We begin by performing a partial fraction expansion for \( Y \) as follows:

\[
Y = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} + \frac{D}{(s-j10)^2} + \frac{E}{s-j10} + \cdots + \frac{F}{(s+\frac{1}{2} - j\frac{\sqrt{3}}{2})^3} + \frac{G}{(s+\frac{1}{2} - j\frac{\sqrt{3}}{2})^2} + \frac{H}{s + \frac{1}{2} - j\frac{\sqrt{3}}{2}} + \cdots \tag{2.192}
\]

From this, we can now immediately write

\[
y(t) = \left[ At + B + Ce^{-t} + 2|D|t \cos(10t + \angle D) + 2|E| \cos(10t + \angle E) \right] 1(t) \\
+ \left[ 2|F|e^{-\frac{1}{2}t} \cos \left(\frac{\sqrt{3}}{2} t + \angle F \right) \right] 1(t) + 2|G|e^{-\frac{1}{2}t} \cos \left(\frac{\sqrt{3}}{2} t + \angle G \right) + 2|H|e^{-\frac{1}{2}t} \cos \left(\frac{\sqrt{3}}{2} t + \angle H \right) \right] 1(t) \tag{2.193}
\]

This is, by far, the most important step. Expressions for the coefficients are then easily written as follows:

\[
A = \lim_{s \to 0} s^2 Y(s) = \frac{1}{[s+1]((s^2 + 100)^2(s^2 + s + 1)^3]} \bigg|_{s=0} = 10^{-4} \tag{2.194}
\]

\[
B = \lim_{s \to 0} \frac{d}{ds} \left[ s^2 Y(s) \right] \tag{2.195}
\]

\[
C = \lim_{s \to -1} (s+1)Y(s) = \frac{1}{s^2(s^2 + 100)^2(s^2 + s + 1)^3]} \bigg|_{s=-1} = \frac{1}{101^2} = 9.80296 \times 10^{-5} \tag{2.196}
\]

\[
D = \lim_{s \to j10} (s-j10)^2 Y(s) = \frac{1}{s^2(s+1)(s+j10)^2(s^2 + s + 1)^3]} \bigg|_{s=j10} = 2.5250 \times 10^{-12} e^{j113.01^\circ} \tag{2.197}
\]

\[
E = \lim_{s \to j10} \frac{d}{ds} \left[ (s-j10)^2 Y(s) \right] \tag{2.198}
\]

\[
F = \lim_{s \to \frac{1}{2} + j\frac{\sqrt{3}}{2}} \left( s + \frac{1}{2} - j\frac{\sqrt{3}}{2} \right)^4 Y(s) = 1.9437 \times 10^{-5} e^{j151^\circ} \tag{2.199}
\]

\[
G = \lim_{s \to \frac{1}{2} + j\frac{\sqrt{3}}{2}} \frac{d}{ds} \left( s + \frac{1}{2} - j\frac{\sqrt{3}}{2} \right)^4 Y(s) \tag{2.200}
\]

\[
H = \lim_{s \to \frac{1}{2} + j\frac{\sqrt{3}}{2}} \frac{1}{2!} \frac{d^2}{ds^2} \left( s + \frac{1}{2} - j\frac{\sqrt{3}}{2} \right)^4 Y(s) \tag{2.201}
\]

The “important” coefficients are \( A \) and \( D \). This is because

\[
y(t) \approx At + 2|D| t \cos(10t + \angle D) \tag{2.202}
\]

after a long time. The coefficients \( A \), \( C \), \( D \), and \( F \) are fairly easy to compute. The other coefficients require more work.

The following MATLAB command sequence was used to determine the coefficients \( A \), \( C \), \( D \), and \( F \) accurately. The other coefficients are more difficult to obtain. Determining the coefficients using MATLAB’s residue command will yield inaccurate results (see explanation in macro below). A symbolic language (e.g. Maple, Mathematica) would be useful to perform the required differentiation. The result could then be used to accurately determine all of the coefficients.
Chapter 2: An Introduction To Laplace Transforms

\[ F = \frac{1}{(s^2 + (s+1) \cdot (s^2 + 100)^2 \cdot (s + 0.5 + j \cdot 0.5 \cdot \sqrt{3})^3)} \]

\[ \text{magF} = \text{abs}(F) \% \text{Determine F accurately} \]
\[ \text{angF} = \text{angle}(F) \cdot 180 / \pi \]

\% Warning: The residue command is very sensitive!
\% Reason: Repeated roots cause the partial fraction expansion problem to be
\% Bottoms line: Be careful using the residue command!!!
\%
\% The following command sequence is provided so that you can compare the
\% accurate answers obtained for A, C, D, F using the above command sequence
\% with the inaccurate results obtained using the residue command.
\%
\% num = [1]; \% Form numerator
den = conv([1 0 0],[1 1]); \% Form s^2 (s+1)
den = conv(den,[1 0 100]); \% Form s^2 (s+1) (s^2 + 100)
den = conv(den,[1 0 100]); \% Form s^2 (s+1) (s^2 + 100)^2
\%
\% Inverse Laplace Transform: Bromwich Integral. Inverse Laplace transforms can (more generally) be
found using the so-called Bromwich integral
\[ f(t) = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} F(s) e^{st} \, ds \quad (2.203) \]
where \( \gamma \) is a real scalar that describes a vertical line \( \text{Re} \, s = \gamma \) in the complex \( s \)-plane that is selected such that all \textit{singularities} (e.g. poles) of \( F \) lie to the left of the vertical line. Using this formula requires advanced concepts from complex variable theory [71], [285]. As such, the above formula will not receive much attention within our modern neo-classical presentation.

2.4 Solution of Linear Ordinary Differential Equations with Constant Coefficients

In this section, we consider the solution of linear ordinary differential equations with constant coefficients - via Laplace transform techniques. Various examples are presented.

Example 2.4.1 (First Order Ordinary Differential Equation)
Consider the following linear ordinary differential equations with constant coefficients
\[ y + y = u \quad (2.204) \]
where \( y(0^-) = y_0 \) and \( u(t) = 10 + \cos t \). Solve for \( y \) for \( t > 0 \).

\% Solution Taking the Laplace transform of each side of the differential equation yields
\[ sY - y_0 + Y = U. \quad (2.205) \]
Solving for \( Y \) then yields
\[ Y(s) = \frac{y_0}{s + 1} + H(s)U(s) \quad (2.206) \]
where
is called the transfer function associated with the differential equation. \( H \) should be viewed as an equivalent representation for the differential equation. The transfer function concept is very useful in studying dynamical system described by linear ordinary differential equations with constant coefficients.

Noting that

\[
U(s) = \frac{10}{s} + \frac{s}{s^2 + 1} = \frac{10(s^2 + 1) + s^2}{s(s^2 + 1)} = \frac{11s^2 + 10}{s(s^2 + 1)}
\]

then yields

\[
Y(s) = \frac{y_o}{s + 1} + \frac{11s^2 + 10}{s(s + 1)(s^2 + 1)}
\]

Following standard partial fraction expansion rules, we obtain

\[
Y(s) = \frac{y_o}{s + 1} + \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s - j1} + \ast.
\]

From this, it follows that

\[
y(t) = y_o e^{-t} + A + Be^{-t} + 2|C| \cos(t + \angle C) \quad \text{for} \quad t > 0.
\]

The coefficients \( A, B, \) and \( C \) can now be computed using standard partial fraction expansion coefficient formulae, as follows

\[
A = \lim_{s \to 0} sH(s)U(s) = \lim_{s \to 0} sH(s) \left[\frac{10}{s} + \frac{s}{s^2 + 1}\right] = \lim_{s \to 0} H(s) \left[\frac{10 + s^2}{s^2 + 1}\right] = 10H(0) = 10
\]

\[
B = \lim_{s \to -1} (s + 1)H(s)U(s) = \lim_{s \to -1} \frac{11s^2 + 10}{s(s^2 + 1)} = -10.5
\]

\[
C = \lim_{s \to j1} (s - j1)H(s)U(s) = \lim_{s \to j1} \frac{11s^2 + 10}{s(s^2 + 1)}
\]

\[
= \lim_{s \to j1} H(s) \frac{11s^2 + 10}{s(s + j1)} = \frac{H(j1)}{2} = \frac{1}{2} |H(j1)| e^{j\angle H(j1)} = \frac{1}{2} \sqrt{2} e^{-j45^\circ}.
\]

It thus follows that

\[
y(t) = y_o e^{-t} + 10H(0) + Be^{-t} + |H(j1)| \cos(t + \angle H(j1))
\]

\[
y(t) = y_o e^{-t} + 10 - 10.5e^{-t} + \frac{1}{\sqrt{2}} \cos(t - 45^\circ)
\]

for \( t > 0. \)

**Exercise 2.4.1 (Differential Equation, Transfer Function, Poles/Zeros, Stability, Steps)**

For each of the following differential equations, determine the system (1) transfer function, (2) poles, (3) zeros (counting zeros at \( \infty \)), (4) stability properties, (5) dc gain, (6) steady state response to a unit step.

\[
\begin{align*}
(a) \quad & \ddot{y} = 6u \\
(b) \quad & \ddot{y} + 10y = 5u \\
(c) \quad & \dot{y} + y = u - \dot{u} \\
(d) \quad & \dot{y} - 2y = u \\
(e) \quad & \ddot{y} = 3u \\
(f) \quad & \ddot{y} + 2\dot{y} + y = (5\dot{u} + 10u) \\
(g) \quad & \ddot{y} + 2\dot{y} + y = 2u \\
h \quad & \dot{y} + 100\ddot{y} = 2u \\
i \quad & \ddot{y} + 9 = 90u \\
j \quad & \ddot{y} + y + y = \dot{u} + 100u \\
k \quad & \dot{y} - 2\dot{y} + 2y = 2u \\
l \quad & \dddot{y} - y + y = \dot{u} + 100u \\
m \quad & \dddot{y} + 2\ddot{y} - 2y = 2u \\
n \quad & \dddot{y} + \dot{y} - y = \dot{u} + 100u \\
o \quad & \frac{d^2y}{dt^2} + 3\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = \dot{u} + 100u \\
p \quad & \frac{d^2y}{dt^2} + 101\frac{d^2y}{dt^2} + 101\frac{dy}{dt} + 100y = \dot{u} + 0.2\ddot{u} + 1.01u \\
q \quad & \dot{y} + 400y = u(t - 0.1)
\end{align*}
\]
Example 2.4.2 (Linearization: From Nonlinear to Linear Models)

In practice, virtually all systems and processes are described by nonlinear differential equations. Linear differential equations are typically approximations to nonlinear differential equations.  

- A vehicle with mass \( m \), velocity \( v \), aerodynamic force \( \beta v^2 \), and applied force \( u \), for example, may be represented (approximated) by the following nonlinear differential equation:

\[
m \dot{v} = -\beta v^2 + u. \tag{2.218}
\]

- An inverted pendulum with length \( l \), mass \( m \) concentrated at tip, angle \( \theta \) with vertical, and applied torque \( u \) at the base may be represented (approximated) by the following nonlinear differential equation:

\[
ml^2 \ddot{\theta} = mgl \sin \theta + u. \tag{2.219}
\]

Even simple nonlinear differential equations such as those above may be cumbersome to work with. Often, such equations are linearized about a natural equilibrium in order to obtain a simpler linear model that can be used to gain insight and for design.

- If the above nonlinear vehicle model is linearized about the equilibrium \( u_e = \beta v_e^2 \) (found by setting \( \dot{v} = 0 \)), one obtains the following linear (small signal) model:

\[
ml \dot{\delta v} = -2\beta v_e \delta v + \delta u \tag{2.220}
\]

where \( \delta v \) will approximate the output of the nonlinear model when \( u = u_e + \delta u \) is applied to the nonlinear model (with initial condition \( v_e \)) and \( \delta u \) is applied to the linear model. Such a linear model (approximation) is expected to be a good approximation to the original nonlinear model if and only if the velocity \( v \) produced by the nonlinear model is sufficiently close to \( v_e \). This will be the case if \( \delta u \) is sufficiently small - so that \( u_e \) is close to \( u_e \). See Exercises ??, ?? for additional discussion on linearization of the car’s translational dynamics.

- If the above nonlinear inverted pendulum model is linearized about the equilibrium \( \theta_e \) (found by setting \( \dot{\theta} = 0 \)), one obtains the following linear (small signal) model:

\[
ml^2 \dot{\theta} = mgl \cos \theta_e \delta \theta + \delta u \tag{2.221}
\]

where \( \delta \theta \) will approximate the output of the nonlinear model when \( u = u_e + \delta u \) is applied to the nonlinear model (with initial condition \( \theta_e \)) and \( \delta u \) is applied to the linear model. Such a linear model (approximation) is expected to be a good approximation to the original nonlinear model if and only if the pendulum angle \( \theta \) produced by the nonlinear model is sufficiently close to \( \theta_e \). This will be the case if \( \delta u \) is sufficiently small - so that \( u_e \) is close to \( u_e \).

Such linear models are often used for system analysis and for control system design. The nonlinear models are often used to (1) corroborate the analysis, (2) to understand under what conditions the analysis is valid, or (3) to evaluate a control system design.

The following example is intended to illustrate how approximations may be used to gain important insight and significantly reduce tedious work.

---

4One of my graduate advisors, Professor Michael Athans, taught me the following about models:

Models have limitations. Stupidity does not!

This very accurate characterization conveys one of the most important ideas that we wish all students and practitioners to acknowledge regarding models and uncertainty.
Consider the following linear ordinary differential equations with constant coefficients

\[ \ddot{y} + 1001 \dot{y} + 1000y = 1000u \]  \hspace{1cm} (2.222)

where \( y(0^-) = y_o, \dot{y}(0^-) = \dot{y}_o, \) and \( u(t) = 10 + \cos t. \)

(a) Relate \( Y(s) \) to \( y_o, \dot{y}_o, \) and \( U(s). \)

(b) Approximate \( y \) for \( t > 0 \). Show how the answer is close to that obtained in Example 2.4.1.

Solution

(a) \( Y(s) = \frac{sy_o + \dot{y}_o + 1001y_o}{(s + 1)(s + 1000)} + H(s)U(s) \) where \( H(s) = \left[ \begin{array}{c} 1 \\ 1000 \\ s + 1000 \end{array} \right] \) and \( U(s) \) yields

\[ Y(s) = \frac{sy_o + \dot{y}_o + 1001y_o}{(s + 1)(s + 1000)} + \frac{1000}{s + 1000} + \frac{10}{s} + \frac{s}{s^2 + 1} \]  \hspace{1cm} (2.223)

\[ \approx \frac{y_o}{s + 1} + \frac{\dot{y}_o}{s + 1000} + \frac{1}{s + 1} \left[ \frac{1000}{s + 1000} + \frac{10}{s} + \frac{s}{s^2 + 1} \right] \]  \hspace{1cm} (2.224)

\[ \approx \frac{y_o}{s + 1} + \frac{\dot{y}_o}{s + 1000} + \frac{A}{s + 1} + \frac{B}{s + 1000} + \frac{C}{s} + \frac{D}{s + 1} + \frac{E}{s + 1000} + \frac{F}{s - 11} + * \]  \hspace{1cm} (2.225)

From this, it follows that

\[ y(t) = y_o e^{-t} + Ae^{-t} + Be^{-1000t} + C + De^{-t} + Ee^{-1000t} + 2[F] \cos(t + \angle F) \]  \hspace{1cm} (2.226)

for \( t > 0. \) The coefficients \( A, B, C, D, E, \) and \( F \) are found using standard simple partial fraction expansion formulae as follows.

\[ A = \left. \frac{y_o + \dot{y}_o}{s + 1000} \right|_{s=-1} \approx \frac{y_o + \dot{y}_o}{1000} \]  \hspace{1cm} (2.227)

\[ B = \left. \frac{y_o + \dot{y}_o}{s + 1000} \right|_{s=-1000} \approx \frac{y_o + \dot{y}_o}{-1000} \]  \hspace{1cm} (2.228)

\[ C = 10H(0) = 10 \]  \hspace{1cm} (2.229)

\[ D = \left. \frac{1000}{s + 1000} \right|_{s=-1000} \approx \frac{11s^2 + 10}{s(s^2 + 1)} \approx -10.5 \]  \hspace{1cm} (2.230)

\[ E = \left. \frac{11s^2 + 10}{s(s^2 + 1)} \right|_{s=-1000} \approx \frac{-11}{1000} \]  \hspace{1cm} (2.231)

\[ F = \frac{H(j1)}{2} = \frac{|H(j1)| e^{\angle H(j1)}}{2} \approx \frac{1}{2} \frac{1}{\sqrt{2}} e^{-j45^\circ} \]  \hspace{1cm} (2.232)

From the above, it follows that

\[ y(t) \approx y_o e^{-t} + \frac{y_o + \dot{y}_o}{1000} e^{-1000t} + 10H(0) + De^{-t} + Ee^{-1000t} + |H(j1)| \cos(t + \angle H(j1)) \]  \hspace{1cm} (2.233)

\[ \approx y_o e^{-t} + \frac{y_o + \dot{y}_o}{1000} e^{-1000t} + 10 - 10.5e^{-t} - \frac{11}{1000} e^{-1000t} + \frac{1}{\sqrt{2}} \cos(t - 45^\circ) \]  \hspace{1cm} (2.234)

which is very close to what was obtained in Example 2.4.1. Given this, we must realize that we did far too much work in this example! We should have made the following reasonable approximation

\[ Y(s) \approx \frac{y_o}{s + 1} + \frac{y_o + \dot{y}_o}{(s + 1)(1000)} + \left[ \frac{10}{s} + \frac{s}{s^2 + 1} \right] \]  \hspace{1cm} (2.235)

Doing so would have saved us a lot of tedious work. The key observation to make in this example is that the transfer function \( H \) may be approximated as follows:

\[ H(s) = \frac{1}{s + 1} \]  \hspace{1cm} (2.236)
for $|s| < 1000$. This implies that our second order differential equation may be approximated by the following first order linear ordinary differential equation
\[
y + y = u
\]
(2.238)
for the input $u$ being considered. This approximation may be justified using the following two facts:
- the pole at $s = -1000$ is much faster than the pole at $s = -1$. Relatively speaking, therefore, the pole at $s = -1000$ has an instantaneous response;
- the poles associated with the input $u$ ($s = 0$ and $s = \pm j1$) are much smaller in magnitude than $1000$.

Making this approximation results in a $y$ which approximates that obtained in Example 2.4.1.

**Example 2.4.4 (Second Order Linear Ordinary Differential Equation)**
Consider the following linear ordinary differential equations with constant coefficients
\[
\ddot{y} + \dot{y} + y = u
\]
where $y(0^-) = y_o$, $\dot{y}(0^-) = \dot{y}_o$, and $u(t) = 10 + \cos t$.

(a) Determine $Y(s)$ in terms of $y(0^-) = y_o$, $\dot{y}(0^-) = \dot{y}_o$, and $U(s)$.
(b) Assuming zero initial conditions, determine $y$ for $t > 0$.
(c) What is the steady state solution $y_{ss}$ for the conditions in (b).

**Solution**
(a) $Y(s) = \frac{y_o + \dot{y}_o s + \frac{1}{s}}{s^2 + s + 1} + H(s) U(s)$ where $H(s) = \frac{1}{s^2 + s + 1}$.
(b) First note that $U(s) = \frac{10}{s} + \frac{s}{s^2 + 1} = \frac{11s^2 + 10}{s(s^2 + 1)}$. Given that we have zero initial conditions, it follows that
\[
Y(s) = H(s) U(s) = \left[ \frac{1}{s^2 + s + 1} \right] \frac{11s^2 + 10}{s(s^2 + 1)} = \frac{A}{s} + \frac{B}{s + \frac{1}{2} - j\frac{\sqrt{3}}{2}} + \frac{C}{s - j1} + *.
\]
(2.240)
Taking inverse transforms - term-by-term, then yields
\[
y(t) = A + 2|B| e^{-\frac{t}{2}} \cos \left(\frac{\sqrt{3}}{2} t + \angle B\right) + 2|C| \cos (t + \angle C).
\]
(2.241)
The coefficients $A$, $B$, $C$ may be determined using standard partial fraction expansion formulae as follows:
\[
A = 10H(0) = 10
\]
(2.242)
\[
B = \left. \frac{1}{s + \frac{1}{2} + j\frac{\sqrt{3}}{2}} \right|_{s = \frac{1}{2} - j\frac{\sqrt{3}}}{11s^2 + 10} = -5 + j3.4641 = 6.0828e^{j145.29^\circ}
\]
(2.243)
\[
C = \frac{H(j1)}{2} = \frac{1}{2} |H(j1)| e^{j\angle H(j1)} = \frac{1}{2} e^{-j90^\circ}.
\]
(2.244)
The coefficient $B$ was computed using MATLAB. From this, it follows that
\[
y(t) = 10H(0) + 2|B| e^{-\frac{t}{2}} \cos \left(\frac{\sqrt{3}}{2} t + \angle B\right) + |H(j1)| \cos (t + \angle H(j1))
\]
(2.245)
\[
= 10 + 12.1656e^{-\frac{t}{2}} \cos \left(\frac{\sqrt{3}}{2} t + 145.29^\circ\right) + \cos (t - 90^\circ).
\]
(2.246)
Note that $y$ has associated with it several natural frequencies or modes - some due to $H$, some due to $u$. 

* * *
In general, if a sinusoid — so important that we have assigned it its own name: the linear systems approximation — obtained in Example 2.4.4.\)

\[ y \text{ applied to a stable LTI system } H \]

Note how the transfer function \( H \) possesses a sinusoid of the same frequency — with purely imaginary complex poles at \( s = \pm j1 \).

(c) From (b), we see that the steady state solution \( y_{ss} \) is given by

\[ y_{ss} = 10H(0) + |H(j1)| \cos(t + \angle H(j1)) = 10 + \cos(t - 90^\circ). \quad (2.247) \]

For additional information, see Section ??, Equation (??), page ??.

The following exercise addresses approximation issues.

**Exercise 2.4.2 (Third Order Ordinary Differential Equation: Approximation)**

Consider the following linear ordinary differential equations with constant coefficients

\[ \begin{align*}
\frac{d^3 y}{dt^3} + 1001 \frac{d^2 y}{dt^2} + 1001 \frac{dy}{dt} + 1000y &= 100u
\end{align*} \quad (2.250) \]

where \( u(t) = 10 + \cos t \). Assuming zero initial conditions, approximate \( y \) for \( t > 0 \). Show that the resulting \( y \) approximates that obtained in Example 2.4.4.

## 2.5 Laplace Transform Problems

This section contains problems on Laplace transforms.

**Problem 2.5.1 (Time and Frequency Shift Properties)**

Verify the following Laplace Transform properties:

\[ a) \mathcal{L}(f(t-T)) = e^{-st} \mathcal{L}(f(t)) = e^{-st} F(s) \quad b) \mathcal{L}(e^{-at}f(t)) = F(s+a) \]

**Problem 2.5.2 (Properties, Initial and Final Value Theorems)**

\[ a) \text{Given that } F(s) = \mathcal{L}(f(t)) = \frac{\omega_o}{s^2 + \omega_o^2} \text{ is the transform of } f(t) = \sin \omega_o t \quad 1(t), \text{ use a suitable Laplace transform property to show that } G(s) = \mathcal{L}(g(t)) = \frac{s}{s^2 + \omega_o^2} \text{ is the transform of } g(t) = \cos \omega_o t \quad 1(t). \]

\[ b) \text{Use (if applicable) the initial value and final value theorems to determine } f(0^+) \text{, } f(\infty) \text{, } g(0^+) \text{, and } g(\infty). \quad \text{Answer: } f(0^+) = 0 \text{, Cannot use final value theorem to determine } f(\infty) \text{, } g(0^+) = 1. \]
**Problem 2.5.3 (Function Decomposition)**

Find the Laplace transform of the function \( f \) shown in Figure 2.2 (without directly evaluating the Laplace integral). Rely strictly on the properties of the transform and assume only that \( \mathcal{L}(1(t)) = 1/s \) is known (i.e., the Laplace transform of the unit step function). Hint: Decompose \( f \) appropriately.

![Figure 2.2: Doublet Pulse-Like Function \( f \) for Problem 2.5.3](image)

**Answers:**
(a) \( \frac{11}{3}e^{-t} - \frac{5}{2}e^{-2t} - \frac{1}{6}e^{-4t}, \ t \geq 0 \)
(b) \( \frac{1}{3}e^{-t} - \frac{1}{6}e^{-2t} \)

**Problem 2.5.4 (Inverse Laplace Transform)**

Find the inverse Laplace transform of the following \( s \)-domain functions: (a) \( Y(s) = \frac{s^2 + 9s + 19}{(s + 1)(s + 2)(s + 4)} \), (b) \( Y(s) = \frac{s}{(s + 1)^2} \).

**Answers:**
(a) \( y(t) = \frac{11}{3}e^{-t} - \frac{5}{2}e^{-2t} - \frac{1}{6}e^{-4t}, \ t \geq 0 \)
(b) \( y(t) = \delta(t) - te^{-t}, \ t \geq 0 \)

**Problem 2.5.5 (Solving Differential Equations)**

Solve \( \ddot{y} + 5\dot{y} + 4y = u \) for \( t \geq 0 \) by means of the Laplace transform with (a) \( u(t) = e^{-2t} \), (b) \( u(t) = \delta(t) \) and zero initial conditions, (c) \( u(t) = 0, y(0) = 0, \dot{y}(0) = 1 \).

**Answers:**
(a) \( y(t) = \frac{11}{3}e^{-t} - \frac{5}{2}e^{-2t} - \frac{1}{6}e^{-4t} \)
(b) \( y(t) = \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} \)

**Problem 2.5.6 (First Order System)**

(a) Determine the steady state value \( y_{ss} = y(\infty) \) associated with \( Y(s) = \frac{15}{s(s + 3)} \). Hint: Examine first term of partial fraction expansion. Approximately how long does it take for \( y \) to reach its steady state value? Determine \( y \). Sketch \( y \). Show that \( y \) is the step response of the LTI system with transfer function \( H(s) = \frac{15}{s^2 + 3} \). Use this and Simulink to validate your sketch. Partial Answer: \( y = H(0)(1 - e^{-3t}) \) where \( H(0) = 5 \) is referred to as the dc gain of \( H \), settling time \( t_s = 5\tau = \frac{5}{3} \).

(b) Solve \( \ddot{y} + 3\dot{y} + 15u = 15u \) for \( u = 1 + \sin t, \ t > 0 \). Show that as \( t \) approaches \( \infty \), \( y \) approaches \( y_{ss} = H(0) + [H(j1)\sin(t + \angle H(j1))] \) where \( H = \frac{15}{s^2 + 3} \). Sketch \( y \) approximately. Hints: Substitute \( y_{ss} \) into differential equation. We need to show that \( \frac{1}{\sqrt{10}} \sin(t - \tan^{-1}1) + \frac{3}{\sqrt{10}} \sin(t - \tan^{-1}2) = \sin t \).

Use two methods: (1) trigonometry, (2) complex arithmetic. The latter requires noting that \( H(0) = 5 \) and \( H(j1) = \frac{15}{s^2 + 3} = \frac{15}{\sqrt{10}}e^{-j(\tan^{-1}1/2)} \) and showing that \( \frac{1}{\sqrt{10}}e^{j(t-\tan^{-1}1/2)} + \frac{3}{\sqrt{10}}e^{j(t-\tan^{-1}2)} = e^{jt} \) or equivalently \( e^{-j(\tan^{-1}1/2)} \left[ \frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}} \right] = 1 \). Note: Complex arithmetic approach is much easier! See Appendix ??.

**Problem 2.5.7 (Second Order Overdamped System and Approximation)**

This problem is an extension of Problem 2.5.6. Its purpose is to demonstrate the value of approximations.

(a) Determine the steady state value \( y_{ss} = y(\infty) \) associated with \( Y(s) = \frac{1500}{s(s + 3)(s + 100)} \). Hint: Examine first term of partial fraction expansion. Approximately how long does it take for \( y \) to reach its steady state value? Determine \( y \). Sketch \( y \). Partial Answer: \( y = A + Be^{-3t} + Ce^{-100t} \) where \( A = \frac{1500}{3(100)} = 5 \), \( B = \frac{1500}{3(100)} \bigg|_{s=3} = \frac{1500}{3(100)} \approx -5 \), \( C = \frac{1500}{(s + 3)(s + 100)} \bigg|_{s=-100} = \frac{1500}{(100)(-97)} \approx 0.15 \). Note: \( y \approx 5(1 - e^{-3t}) \).

(b) Show that \( y \) is the step response of the LTI system with transfer function \( H(s) = \frac{1500}{s(s + 3)(s + 100)} \). Use this and Simulink to validate your sketch. Show that your answer here does not differ much from that obtained in Problem 2.5.6. Main Point: We could have approximated \( Y \) as follows: \( Y(s) = \frac{1500}{s(s + 3)(s + 100)} = \frac{1500}{s\cdot 3(100)} \approx \frac{1500}{s(100)} \).
The dynamics of a car, undergoing one-dimensional translation, are given by

\[
\frac{15}{s+3} \begin{bmatrix} 15 \\ 100 \end{bmatrix} = \frac{15}{s+3} \begin{bmatrix} 15 \\ 100 \end{bmatrix} \text{ to get } Y \text{ from Problem 2.5.6.}
\]

(c) Solve \( \ddot{y} + 103\dot{y} + 300y = 1500u \) for \( y \) when \( u = 1 + \sin t, \ t > 0 \). Show that as \( t \) approaches \( \infty \), \( y \) approaches \( y_{ss} = H(0) + |H(j1)| \sin(t + \angle H(j1)) \) where \( H = \frac{1500}{s^2 + 103s + 300} = \frac{15}{s+3} \begin{bmatrix} 15 \\ 100 \end{bmatrix} \). Sketch \( y \) approximately.

\[\text{Problem 2.5.8 (Higher Order System)}\]

(a) Determine the inverse Laplace transform of \( Y(s) = \frac{24}{s(s^2 + 4s + 8)} \). Determine the steady state value \( y(\infty) \). Approximately how long does it take for \( y \) to reach its steady state value? Answer: \( Y = A + \frac{2B}{s}e^{-\beta_1 t} \cos(2t + \angle B) \) where \( A = H(0) = 3, \ B = \frac{24}{s(s^2 + 4s + 8)} \) evaluated at \( s = -2+2j2 \) \( = \frac{-2+2j2}{(2\sqrt{2}e^{j225^\circ})(4e^{j90^\circ})} = \frac{3}{2}e^{-j225^\circ} \), \( y_{ss} = y(\infty) = A = H(0) = 3 \), settling time is \( t_s = 5\tau = \left[ \frac{5}{2\sqrt{2}e^{j225^\circ}} \right] = \frac{5}{2} = 2.5 \text{ sec.} \)

(b) Use calculi to (i) determine the peak value \( y_{\text{max}} \) \( \equiv \max_t y(t) \) and the associated peaking time \( t_p \), (ii) determine the associated overshoot \( M_p \equiv \frac{y_{\text{max}} - y_{ss}}{y_{ss}} \), and (iii) sketch \( y \). Use Simulink to validate your sketch. Hint: See Example ?? for relevant formulae.

(c) Solve \( \ddot{v} + 4\dot{v} + 8v = 24u \) for \( v \) when \( u = 1 + \sin 100t, \ t > 0 \). Show that as \( t \) approaches \( \infty \), \( y \) approaches \( y_{ss} = H(0) + |H(j1)| \sin(100t + \angle H(j100)) \) where \( H = \frac{24}{s^2 + 4s + 8} \). Sketch \( y \) approximately.

\[\text{Problem 2.5.9 (Higher Order System)}\]

Determine the form of the inverse Laplace transform of \( Y(s) = \frac{1000(s^2+0.81)}{s^2(s^2+10s+100)(s^2+1)(s+0.01)^2(s-5)} \). Use MATLAB to determine the coefficients. Show that as \( t \) approaches \( \infty \), \( y \) approaches a function having the form \( y_{ss} = At + B + C\cos(t+D) + Ae^{Mt} \). Show how to determine \( A, B, C, D, E \).

\[\text{Problem 2.5.10 (Analysis of Linearized Translational Dynamics for Car)}\]

In this problem, we examine the linearized translational dynamics for a car. It can be shown that the linearized dynamics of a car, undergoing one-dimensional translation, are given by

\[
\dot{v} = \left( -\frac{2\beta v_e}{m} \right)v + \frac{1}{m}u \quad (2.251)
\]

where \( v \) (measured in \( m/s \)) represents the speed, \( u \) (measured in \( N \)) represents the horizontal force due to the engine, \( m \) (measured in \( kg \)) represents the mass of the car, and \( \beta > 0 \) (measured in \( N/(m/s)^2 \)) represents the aerodynamic drag coefficient of the car. The derivation of the nonlinear model and its linearization are addressed within Exercise ??.

(a) \( s \)-domain Relationship. Assuming zero initial conditions, determine the \( s \)-domain relationship between the speed \( v \) and the horizontal force \( u \) i.e. What is the associated system transfer function?

Answer: \( V(s) = \frac{v_o}{s^2 + \frac{2\beta v_e}{m} + P(s)U(s)}, P(s) \equiv \left. \frac{V(s)}{U(s)} \right|_{v_o=0} = \frac{1}{s + \frac{2\beta v_e}{m}}, v_o = v(0) \) denotes the initial velocity.

(b) Initial Condition Response. Determine \( v \) when \( u = 0 \) and the initial velocity is \( v(0^-) = v_o \). Based upon the unforced or natural response obtained, would you say that this system is stable, unstable, or marginally stable? Explain. Answer: \( v(t) = v_o e^{-\frac{2\beta v_e}{m} t} \) \( t(t) \), stable.

(c) Impulse Response. Determine \( v \) when \( u(t) = \delta(t) \).

Answer: \( v(t) = \frac{1}{m} e^{-\frac{2\beta v_e}{m} t} 1(t) \). Sketch \( v(t) \)? How is the answer modified if \( v(0^-) = v_o \) is non-zero?

(d) Step Response. Determine \( v \) and steady state response \( v_{ss} \) when \( u(t) = 1(t) \). What is the associated time constant? settling time? Answer: \( v(t) = P(0)(1 - e^{-\frac{2\beta v_e}{m} t}) 1(t) = \frac{1}{2\beta v_e}(1 - e^{-\frac{2\beta v_e}{m} t}) 1(t). \) \( P(0) \) is often referred to as the \( dc \) gain of the system. Sketch \( v \). How is \( v \) modified if \( v(0^-) = v_o \) is non-zero?

(e) Response to Sinusoid. Determine \( v \) and steady state response \( v_{ss} \) when \( u(t) = \sin \omega_d t \). Answer: \( v(t) = \left[ \frac{1}{\omega_d^2 + \frac{2\beta v_e}{m} + \frac{1}{m}} \right] e^{-\frac{2\beta v_e}{m} t} + v_{ss} \) \( 1(t) \), \( v_{ss} = |P(j\omega_o)| \sin(\omega_d t + \angle P(j\omega_o)) = \frac{\frac{1}{\omega_d} \omega_o}{\sqrt{\omega_o^2 + \frac{2\beta v_e}{m}}} \sin(\omega_d t - \tan^{-1}\frac{\omega_o}{\frac{2\beta v_e}{m}}) \).
Problem 2.5.11 (Analysis of Linearized Fixed-Base Inverted Pendulum)

In this problem, we examine the linearized dynamics for a fixed-base inverted pendulum. It can be shown that the linear dynamics of a single degree-of-freedom fixed-base inverted pendulum (robotic link) are given by

\[ \ddot{\theta} - \frac{g}{l} \dot{\theta} = \frac{1}{ml^2} u \]  

(2.252)

where \( \theta \) (measured in radians) represents the angle that the pendulum makes with the vertical, \( u \) (measured in Nm) represents the torque applied at the base, \( m \) (measured in kg) represents the mass of the pendulum and is assumed to be concentrated at the end of the link, \( l \) (measured in m) represents the length of the pendulum, and \( g \) (measured in m/s\(^2\)) represents the acceleration due to gravity. The derivation of the nonlinear model and its linearization are addressed within Exercise ??.

(a) s-domain Relationship. Assuming zero initial conditions, determine the s-domain relationship between the pendulum angle \( \theta \) and the applied torque \( u \)? i.e. What is the associated system transfer function?

(b) Initial Condition Response. Determine \( \theta \) when \( u = 0 \) and the initial velocity is \( \dot{\theta}(0^-) = \theta_o \)? Based upon the unforced or natural response obtained, would you say that this system is stable, unstable, or marginally stable? Explain. How would your answer change if \( \dot{\theta}(0^-) = \theta_o \) is non-zero?

(c) Impulse Response. Determine \( \theta \) when \( u(t) = \delta(t) \). Sketch \( \theta \). How is answer modified if \( \dot{\theta}(0^-) = \theta_o \) is non-zero?

(d) Step Response. Determine \( \theta \) and steady state response \( \theta_{ss} \) when \( u(t) = 1(t) \). Sketch \( \theta \). How is answer modified if \( \dot{\theta}(0^-) = \theta_o \) is non-zero?

(e) Response to Sinusoid. Determine \( \theta \) and steady state response \( \theta_{ss} \) when \( u(t) = \sin \omega_o t \).

Problem 2.5.12 (Analysis of Aircraft Pitch Dynamics)

In this problem, we examine the simplified linear pitch attitude dynamics for a fixed-wing aircraft. It can be shown that the linear pitch dynamics of an aircraft can be approximated as follows:

\[ I \ddot{\theta} + b \dot{\theta} + k_1 \theta = k_2 (l_1 + l_2) u \]  

(2.253)

where \( \theta \) represents the pitch angle (measured in radians) that the aircraft makes with the horizontal, \( u \) (measured in radians) represents the elevator deflection, \( I > 0 \) (measured in kgm\(^2\)) represents the moment of inertia of the aircraft about the pitch axis, \( b > 0 \) (measured in Nm/sec) represents a rotational pitch damping coefficient, \( l_1 \) (measured in m) represents the distance which the c.p. (center of pressure) lies aft of the c.g. (center of gravity), \( l_2 > 0 \) (measured in m) represents the distance from the elevator to the c.p., \( k_1 > 0 \) (measured in N/sec) represents a lift coefficient, and \( k_2 > 0 \) (measured in N) represents the elevator effectiveness coefficient. Note that \( l_1 \) can be positive, negative, or zero. The derivation of this model is addressed within Exercise ??.

For simplicity, assume that the moment of inertia of the aircraft is negligible (i.e. \( I \approx 0 \)).

(a) s-domain Relationship. Assuming zero initial conditions, determine the s-domain relationship between the pitch attitude \( \theta \) and the elevator deflection \( u \)? i.e. What is the associated system transfer function?

(b) Initial Condition Response. Determine \( \theta \) when \( u = 0 \) and the initial velocity is \( \dot{\theta}(0^-) = \theta_o \)? Suppose that \( l_1 > 0 \). Based upon the unforced or natural response obtained, would you say that this system is stable, unstable, or marginally stable? Explain. How does your answer change if \( l_1 < 0 \) ? \( l_1 = 0 \)?

(c) Impulse Response. Determine \( \theta \) when \( u(t) = \delta(t) \). Sketch \( \theta \). How is answer modified if \( \dot{\theta}(0^-) = \theta_o \) is non-zero?

(d) Step Response. Determine \( \theta \) when \( u(t) = 1(t) \). Sketch \( \theta \). How is answer modified if \( \dot{\theta}(0^-) = \theta_o \) is non-zero?

(e) Response to Sinusoid. Determine \( \theta \) and steady state response \( \theta_{ss} \) when \( u(t) = \sin \omega_o t \).
Problem 2.5.13 (Analysis of Simple Satellite Dynamics)
In this problem, we examine the simplified linear pitch axis dynamics of a satellite. It can be shown that these dynamics can be approximated by

\[ I \ddot{\theta} = bu + d \]  \hspace{1cm} (2.254)

where \( \theta \) (measured in radians) represents the pitch attitude of the satellite, \( u \) (measured in N) represents a force input produced by a reaction jet, \( d \) (measured in Nm) represents a disturbance torque, \( I > 0 \) (measured in kgm\(^2\)) represents the moment of inertia about the pitch axis, and \( b > 0 \) (measured in m) represents the moment arm associated with \( u \) about the pitch axis. The derivation of this model is addressed within Exercise ??.

Assume that the disturbance is negligible (i.e. \( d \approx 0 \)).

(a) s-domain Relationship. Assuming zero initial conditions, determine the s-domain relationship between the pitch attitude \( \theta \) and the applied input force \( u \); i.e. What is the associated system transfer function?

(b) Initial Condition Response. Determine \( \theta \) when \( u = 0 \) and \( \theta(0^-) = \theta_o \). Based upon the unforced or natural response obtained, would you say that this system is stable, unstable, or marginally stable? Explain.

(c) Impulse Response. Determine \( \theta \) when \( u(t) = \delta(t) \). Sketch \( \theta(t) \). How is answer modified if \( \theta(0^-) = \theta_o \) is non-zero?

(d) Step Response. Determine \( \theta \) and steady state response \( \theta_{ss} \) when \( u(t) = 1 \). Sketch \( \theta \). How is answer modified if \( \theta(0^-) = \theta_o \) is non-zero?

(e) Response to Sinusoid. Determine \( \theta \) and steady state response \( \theta_{ss} \) when \( u(t) = \sin \omega_o t \).

(f) Repeat (a) - (e) where pitch angular velocity \( \omega \) is the desired output. 

Hint: Define \( \omega \) def = \( \dot{\theta} \). Then the satellite dynamics can be written as \( I \dot{\omega} = bu \).

Problem 2.5.14 (Analysis of Simple Thermal Dynamics for Insulated Stirred Tank)
In this problem, we examine the thermal dynamics for a substance within an insulated and perfectly stirred tank with an inlet and outlet. It can be shown that the simplified heat transfer dynamics are given by

\[ RC\dot{y} + y = u \]  \hspace{1cm} (2.255)

where \( y \) represents the temperature (measured in \(^\circ C\)) of a substance flowing out of the tank, \( u \) represents the temperature (measured in \(^\circ C\)) of the substance flowing into the tank, \( R > 0 \) represents the thermal resistance of the substance, and \( C > 0 \) represents the thermal capacitance of the substance. It is assumed that (1) the tank is insulated so that there is no heat loss, (2) there is no heat storage within the system so that the heat out is equal to the heat in, and (3) that the substance within the tank is perfectly mixed (stirred tank assumption) so that its contents are at a uniform temperature.

(a) s-domain Relationship. Assuming zero initial conditions, determine the s-domain relationship between the output temperature \( y \) and the input temperature \( u \); i.e. What is the associated system transfer function?

(b) Initial Condition Response. Determine \( y \) when \( u = 0 \) and the initial temperature is \( y(0^-) = y_o \). Based upon the unforced or natural response obtained, would you say that this system is stable, unstable, or marginally stable? Explain.

(c) Impulse Response. Determine \( y \) when \( u(t) = \delta(t) \). Sketch \( y(t) \). How is answer modified if \( y(0^-) = y_o \) is non-zero?

(d) Step Response. Determine \( y \) and steady state response \( y_{ss} \) when \( u(t) = 1 \). Sketch \( y \). How is answer modified if \( y(0^-) = y_o \) is non-zero?

(e) Response to Sinusoid. Determine \( y \) and steady state response \( y_{ss} \) when \( u(t) = \sin \omega_o t \).
2.6 Summary and Conclusions

This chapter has provided an overview of Laplace transforms. While their utility is much broader, Laplace transforms are particularly useful for analyzing continuous-time signals and continuous-time dynamical systems that can be modeled or approximated by linear time invariant (LTI) systems described by linear ordinary differential equations with constant coefficients.