A Practical Approach To Signals, Systems, and Controls

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September 1, 1998
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Chapter 1

Introduction: Systems And The Use Of Mathematical Models

1.1 Widespread Applicability And Common Features Of Mathematical Models

With the advent of powerful computers, the use of mathematical models by mathematicians, scientists, engineers, and researchers has become commonplace. Mathematical models are used for analysis, design, and to predict the behavior of complex

- **physical systems** (e.g. airplanes, robots, submarines, economic systems, musculoskeletal systems, heat exchangers, satellites, electrical networks, weather systems, missile target intercepts, etc.),

- physical, social, economic, political, and virtual **processes** (e.g. semiconductor manufacturing processes, chemical processes, thermal diffusion, breathing, combustion, signal processing, population growth, securities trading, supply and demand, decision making, battle management, etc.), and

- physical, social, economic, and political **phenomena** (e.g. electrical, mechanical, thermal, optical, inflation, interest rates, structural vibration and deformation, ozone layer depletion, etc.).

*Common Features.* While the above list is intended to illustrate the widespread applicability of mathematical models, it is more important to note three common features which each element of
the list possesses. These common features include:

- **input variables** responsible for causing certain actions, behaviors, or phenomena to take place (e.g. fuel flow to engine, torque applied to robotic manipulator arm, etc.),

- **output variables** which are associated with some effect, action, or phenomena (e.g. speed of aircraft, angle of robotic manipulator arm, etc.), and

- an implicit, possibly unknown or uncertain, cause-effect relationship between input and output variables - each depending on some independent dynamic (changing) variable such as time.

*Internal variables* which are neither inputs nor outputs, or which are associated with energy (also called *state variables*), may also be involved.

### 1.2 A Unifying Framework: The Concept Of A Dynamical System

These common features provide a unifying framework which enables the systematic and uniform treatment of diverse subject matter which may appear unrelated (see above list). More specifically, they motivate the concept of a dynamical system or system - a concept which was developed to refer to the implicit cause-effect relationship between input variables internal variables, and output variables.

The concept of a system generalizes the simple notion of a function to dynamical settings. To understand this, it is useful to recall the definition of a function:

- Let $U$ and $Y$ denote two nonempty sets. A function $T(\cdot)$ from $U$ to $Y$ is a rule of correspondence that assigns exactly one element of $Y$ to each element of $U$. The first set $U$ is called the domain of $T(\cdot)$. The second set $Y$ is called the co-domain of $T(\cdot)$.

For systems, the “system” represents the “function” and the natural elements of the “function domain” $U$ and “co-domain” $Y$ are themselves functions [50].
1.3 Linear Systems: A Good Tool For Approximation

Linear dynamical systems or linear systems represent a very large class of systems, or mathematical models, which are used to describe, model, simulate, emulate, or approximate systems exhibiting “linear behavior.” Recall that a linear relationship $T(\cdot)$ is one which satisfies the following property

$$T(a_1 u_1 + a_2 u_2) = a_1 T(u_1) + a_2 T(u_2)$$

for all scalars $a_1, a_2$ and all “input elements” $u_1, u_2$ within the domain of $T(\cdot)$.

Linear systems are used in all fields of science and engineering (e.g. robotics, semiconductor manufacturing, missile navigation, guidance and control, economics, etc.). While the use of such models is widespread, it is important to note that they are typically used as approximations to more complex nonlinear systems or models. As such, users of system models, and especially linear models, must keep in mind the following compelling truth: “mathematical models have limitations, stupidity does not!” The point here, is that it is very important to understand how well or poorly a mathematical model approximates the behavior of a system.

Connection With Mathematical Objects. More precisely, the term linear system is generally used to refer to systems whose inputs, outputs, and internal variables (e.g. state variables) are related by

1. linear ordinary differential equations, linear algebraic and integral relationships, and linear partial differential equations, and/or
2. linear ordinary difference equations, linear algebraic and series relationships, and linear partial difference equations.

1.4 System Types

Continuous Time Systems. Systems described by linear ordinary differential equations, linear algebraic and integral relationships, and linear partial differential equations, are referred to as continuous time systems since they depend on a continuous (real valued) independent variable $t$ - usually
representing time [10], [32].

Discrete Time Systems. Systems described by linear ordinary difference equations, linear algebraic and series relationships, and linear partial difference equations, are referred to as discrete time systems since they depend on a discrete (integer valued) independent variable \( n \) - usually representing discrete time [33], [34], [35].

Sampled-Data Systems. Systems described by both differential and difference equations are referred to as sampled-data systems [9], [18].

General Outline. In what follows, the initial focus will be placed on continuous time systems described by linear ordinary differential equations with constant coefficients [32]. The development will then shift to discrete time systems described by linear ordinary difference equations with constant coefficients.

Before developing mathematical models for continuous time systems, it is essential to develop mathematical models for typically encountered continuous time test signals (functions).
Chapter 2

Mathematical Preliminaries

2.1 Complex Numbers And Complex Arithmetic

Complex numbers are particularly important in the study of linear dynamical systems. Their main utility lies in the fact that they permit engineers to represent sinusoidal and exponential sinusoidal functions by complex exponentials. Given this, complex exponentials greatly facilitate the study of linear systems. Moreover, their use leads naturally to new powerful system concepts (e.g. frequency response of a system, etc.). As such, complex numbers are indispensable in the study of dynamical systems.

Motivation: The Need For Complex Numbers. Because equations of the form

\[ z^2 = -1 \]  \hspace{1cm} (2.1)

were found to be present everywhere in science and engineering, it became essential to invent complex numbers. (Necessity is the mother of invention!)

Definition 2.1.1 Definition: Complex Numbers.

A complex number \( z \) is an ordered pair of real numbers \( x \) and \( y \). The complex number \( z \) is denoted \( z = (x, y) \). The real number \( x \) is called the real part of the complex number \( z = (x, y) \). The real number \( y \) is called the imaginary part of the complex number \( z = (x, y) \). If \( x = 0 \), we say that the
complex number \( z = (0, y) \) is purely imaginary. If \( y = 0 \), we say that the complex number \( z = (x, 0) \) is real. The set of all complex numbers is denoted \( \mathbb{C} \).

Equality of Complex Numbers. Two complex numbers \( z_1 = (x_1, y_1) \) and \( z_2 = (x_2, y_2) \) are equal if and only if their real parts are equal and their imaginary parts are equal; i.e.

\[
z_1 = z_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2.
\]

(2.2)

Addition of Complex Numbers. The sum of two complex numbers \( z_1 = (x_1, y_1) \) and \( z_2 = (x_2, y_2) \) is denoted \( z_1 + z_2 \) and defined as follows:

\[
z_1 + z_2 \overset{\text{def}}{=} (x_1 + x_2, y_1 + y_2).
\]

(2.3)

That is, the sum is obtained by adding the real parts and by adding the imaginary parts to get the new real and imaginary parts, respectively. This definition implies that complex numbers add like vectors. An intuitive geometric interpretation is thus possible.

Multiplication of Complex Numbers. The product of two complex numbers \( z_1 = (x_1, y_1) \) and \( z_2 = (x_2, y_2) \) is denoted \( z_1 z_2 \) and defined as follows:

\[
z_1 z_2 \overset{\text{def}}{=} (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).
\]

(2.4)

While this definition is easy to use, it is rather strange and arguably unmotivated. An intuitive (geometric) interpretation for complex multiplication will be given below.

\[
\begin{align*}
\text{Commutative, Associative, and Distributive Properties.} & \quad \text{It can be shown that the addition and multiplication of complex numbers are commutative, associative, and distributive operations. That is, for any complex numbers } z_1, z_2, \text{ and } z_3, \text{ the following relationships hold:} \\
z_1 + z_2 &= z_2 + z_1 & z_1 z_2 &= z_2 z_1 \\
(z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) & (z_1 z_2) z_3 &= z_1 (z_2 z_3) \\
z_1 (z_2 + z_3) &= z_1 z_2 + z_1 z_3 & (z_1 + z_2) z_3 &= z_1 z_3 + z_2 z_3.
\end{align*}
\]

(2.5)
Most of us take these properties for granted when working with real numbers. It is very comforting that they hold for complex numbers as well.

The Symbol $j$. Given the above definition for complex multiplication, it follows that $(0,1)(0,1) = (-1,0) = -1$ and $(0,-1)(0,-1) = (0,1)(0,1) = (-1,0) = -1$. From this, it follows that $z = (0,1)$ and $z = (0,-1) = -(0,1)$ are each solutions to the quadratic equation $z^2 = -1$. Because of this, we assign the special symbol $j$ to $(0,1)$; i.e.,

$$j \overset{\text{def}}{=} (0,1).$$

Because $j^2 = -1$, it follows that $j = \sqrt{-1}$. It should be noted that while the symbol $j$ is widely used within the engineering community, the symbol $i$ is still the preferred symbol within the mathematics community [1], [3], [41]. The symbol $i$ is used to represent electrical current within the engineering community [23].

Rectangular Representation. With the above “$j$-notation” in place, it follows that

$$(x, y) = (x, 0) + (0, y)$$

$$(x, 0) + (0,1)(y, 0)$$

$$(x, 0) + j(y, 0)$$

$$x + jy$$

This provides motivation for the widely used notation

$$z = x + jy$$

which we adopt hereafter. This representation for $z$ is commonly referred to as the rectangular representation for $z$. Given this, the complex number $z = x + jy$ can be visualized as indicated in Figure 2.1. The idea of representing complex numbers by points (vectors) in the plane was
formulated by Gauss in his 1799 dissertation and, independently, by Argand in 1806 [3, pg. 17]. Figure 2.1 is sometimes referred to as an Argand diagram.

![Complex z-Plane Diagram](image)

Figure 2.1: Visualizing a Complex Number $z$ In The Complex z-Plane

With the above notation, equations (2.3) and (2.4) become

$$z_1 + z_2 \overset{\text{def}}{=} [x_1 + x_2] + j[y_1 + y_2]$$  \hspace{1cm} (2.12)

$$z_1 z_2 \overset{\text{def}}{=} [x_1 x_2 - y_1 y_2] + j[x_1 y_1 + x_2 y_2].$$  \hspace{1cm} (2.13)

In order to properly define complex division, it is useful to introduce new terms.

**Conjugate of a Complex Number.** The *conjugate* of a complex number $z = x + jy$ is denoted $\bar{z}$ and defined as follows:

$$\bar{z} \overset{\text{def}}{=} x - jy.$$  \hspace{1cm} (2.14)

The conjugate $\bar{z}$ of the complex number $z = x + jy$ may visualized as shown if Figure 2.2. Moreover, it can be shown that

$$\bar{\bar{z}} = z.$$  \hspace{1cm} (2.15)
Magnitude of a Complex Number. The magnitude of a complex number $z = x + jy$ is denoted $|z|$ and defined as follows:

$$|z| \equiv \sqrt{x^2 + y^2}. \quad (2.16)$$

The magnitude $|z|$ can be visualized as shown in Figure 2.2. From the figure, and the Pythagorean theorem, it follows that $|z|$ is precisely the length of the vector used in representing $z$. It is useful to note that

$$|z| = \sqrt{z \overline{z}} \quad (2.17)$$

Division. Given the above, the division of two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ ($z_2 \neq 0$) is defined in terms of the ratio

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} \quad (2.18)$$

as follows

$$\frac{z_1}{z_2} \equiv \left[ \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \right] + j \left[ \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \right]. \quad (2.19)$$
As with complex multiplication, an intuitive (geometric) interpretation for complex division will be given below.

The rectangular representation for complex numbers is most useful for carrying out and visualizing the addition and subtraction of complex numbers. Rectangular representations, however, are not very intuitive when it comes to carrying out and visualizing complex multiplication and division. To obtain nicer formulae - with useful geometric interpretation, we need to motivate an alternative representation for complex numbers. Recalling that real exponentials satisfy very nice relationships for multiplication and division, we are led naturally to polar representations and complex exponentials. To properly introduce these ideas, we require a new term - the angle of a complex number.

**Angle Of A Complex Number.** Consider the complex number \( z = x + jy \) shown in Figure 2.3. The angle of the complex number \( z \) is defined as follows:

\[
\angle z \overset{\text{def}}{=} \begin{cases} 
\tan^{-1} \frac{y}{x} & x > 0, \ y > 0 \\
180 - \tan^{-1} \frac{y}{x} & x < 0, \ y > 0 \\
-180 + \tan^{-1} \frac{|y|}{|x|} & x < 0, \ y < 0 \\
-\tan^{-1} \frac{|y|}{x} & x > 0, \ y < 0 
\end{cases} \tag{2.20}
\]

Engineers often use the word *phase* instead of angle. From Figure 2.3, it follows that the real and imaginary parts of a complex number \( z \) may be readily computed from the magnitude and angle of \( z \) as follows:

\[
Re\{z\} = |z| \cos \angle z \tag{2.21}
\]

\[
Im\{z\} = |z| \sin \angle z. \tag{2.22}
\]

This motivates the alternative representation that we seek for complex numbers - the so-called *polar* or *exponential representation.*
Polar (Exponential) Representation. From the above discussion, it follows that

\[ z = |z|e^{j\angle z} \]  \hspace{1cm} (2.23)

where

\[ e^{j\angle z} \overset{\text{def}}{=} \cos \angle z + j \sin \angle z. \]  \hspace{1cm} (2.24)

Equation (2.23) defines the the polar or exponential representation for the complex number \( z \).

Polar Representation Properties: Geometric Interpretations For Multiplication And Division. The polar representation of a complex number offers very nice properties when it comes to complex multiplication and division. If \( z_1 = |z_1|e^{\angle z_1} \) and \( z_2 = |z_2|e^{\angle z_2} \), then it can be shown that

\[ z_1z_2 = |z_1||z_2|e^{j(\angle z_1 + \angle z_2)} \]  \hspace{1cm} (2.25)

\[ \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\angle z_1 - \angle z_2)}. \]  \hspace{1cm} (2.26)

Unlike their rectangular counterparts (cf. equations (2.13), (2.19)), these properties are very useful and are readily visualized geometrically. Equation (2.25) shows that the multiplication of two complex numbers involves the multiplication of their magnitudes and the addition of their angles.
Equation (2.26) shows that the division of two complex numbers involves the division of their magnitudes and the subtraction of their angles.

**Euler Identities.** From the above discussion and Figure 2.4, one obtains the following Euler identities:

\[
\begin{align*}
    z + \bar{z} &= 2Re\{z\} = 2|z| \cos \angle z \\
    z - \bar{z} &= j2\text{Im}\{z\} = 2|z| \sin \angle z.
\end{align*}
\]

These identities are extensively used in science and engineering.

**Complex Exponential, Periodicity.** Just as the polar representation of a complex number was motivated by the nice multiplication and division properties that exponentials offer, these ideas motivate the definition for the *complex exponential* $e^z$ of a complex number $z = x + jy$:

\[
e^z \overset{\text{def}}{=} e^x e^{jy}
\]

From equation (2.24), it then follows that

\[
e^z = e^x e^{jy} = e^x \cos y + je^x \sin y.
\]
From this, it follows that the magnitude of $e^z$ is given by

$$|e^z| = e^x$$  \hspace{1cm} (2.31)

and the angle $e^z$ is given by

$$\angle e^z = y.$$  \hspace{1cm} (2.32)

It is very important to note that the complex exponential is periodic in the angle $y$ with period $2\pi$. Thus, $e^0$ and $e^{j2\pi}$ are different representations for the same number, namely $z = 1$.

**Roots.** The polar representation of a complex number $z$ is also very useful for determining the roots of the number as follows:

$$z^m = |z|^m e^{j\frac{m\pi}{n} \angle z}$$  \hspace{1cm} (2.33)

$$= |z|^m e^{j\frac{m\pi}{n}(\angle z + 2\pi k)} \quad k = 0, 1, \ldots n - 1.$$  \hspace{1cm} (2.34)

**Example 2.1.1 (Cube Root of -1.)**

Since the number $-1$ may be expressed as $e^{-j180^\circ}$, $e^{j180^\circ}$, and $e^{j360^\circ}$, it follows that the three roots which satisfy $z^3 = -1$ are given by

$$z = e^{-j60^\circ}, e^{j60^\circ}, e^{j180^\circ}.$$  \hspace{1cm} (2.35)

**Example 2.1.2 (Generation of Exponential Sinusoids From Complex Exponentials.)**

One very important use of complex exponentials lies in their utility to construct real exponential sinusoids. This follows from the relationship:

$$x(t) = |X| e^{\sigma t} \cos(\omega_d t + \angle X) = |X| e^{\sigma t} \operatorname{Re}\{e^{j(\omega_d t + \angle X)}\}$$  \hspace{1cm} (2.36)

$$= \operatorname{Re}\{|X| e^{\sigma t} e^{j(\omega_d t + \angle X)}\}$$  \hspace{1cm} (2.37)

$$= \operatorname{Re}\{|X| e^{j\angle X} e^{(\sigma + j\omega_d)t}\}.$$  \hspace{1cm} (2.38)
Similarly,

\[
x(t) = |X|e^{\sigma_0 t} \sin(\omega_0 t + \angle X) = |X|e^{\sigma_0 t} \text{Im}\{e^{j(\omega_0 t + \angle X)}\} = \text{Im}\{|X|e^{\sigma_0 t}e^{j(\omega_0 t + \angle X)}\} = \text{Im}\{|X|e^{j\angle X}e^{(\sigma_0 + j\omega_0)t}\}.
\]

We say that the above exponential sinusoids are generated from the complex exponential

\[
z = |z|e^{j\angle z} = |X|e^{\sigma_0 t}e^{j(\omega_0 t + \angle X)} = |X|e^{j\angle X}e^{(\sigma_0 + j\omega_0)t} = Xe^{s_0 t}
\]

where

\[
s_0 = \sigma_0 + j\omega_0
\]

is called the complex frequency of the exponential sinusoids and

\[
X \overset{\text{def}}{=} |X|e^{j\angle X}
\]

is called the phasor representation for the function \(x(t)\). It is often very useful to associate the above real exponential sinusoids with their phasor representations. This association is made in many areas of science and engineering. One area includes the steady state analysis of linear time invariant (LTI) continuous time dynamical systems \[32\] and the steady state analysis of linear shift invariant (LSI) discrete time dynamical systems \[33\].

\[
\text{Example 2.1.3 (Addition Of Sinusoids Using Complex Phasors.)}
\]

Addition Of Two Sinusoids. Consider the continuous time signal

\[
f(t) = 2\sin(\omega_0 t - 30^\circ) + \cos(\omega_0 t + 45^\circ).
\]

This signal is the sum of two sinusoids of the same frequency \(\omega_0\). The goal in this example is to illustrate how complex phasors may be used to add the sinusoids to get a single cosine signal of frequency \(\omega_0\). We proceed algebraically using the trigonometric relationship

\[
\sin(x) = \cos(x - 90^\circ)
\]
and Euler’s identities, as follows:

\[ f(t) = 2 \cos(\omega_0 t - 120^\circ) + \cos(\omega_0 t + 45^\circ) \]  
\[ = 2 \text{Re}\{e^{j(\omega_0 t - 120^\circ)}\} + \text{Re}\{e^{j(\omega_0 t + 45^\circ)}\} \]  
\[ = \text{Re}\{2e^{-j120^\circ} e^{j\omega_0 t}\} + \text{Re}\{e^{j15^\circ} e^{j\omega_0 t}\} \]  
\[ = \text{Re}\{2e^{-j120^\circ} + e^{j15^\circ}\} \]  
\[ = \text{Re}\{|X| e^{j(\omega_0 t + \angle X)}\} \]  
\[ = |X| \cos(\omega_0 t + \angle X) \]  

where the critical calculation lies in the addition of complex phasors associated with each sinusoid as follows:

\[ X = 2e^{-j120^\circ} + e^{j15^\circ} \]  
\[ = -1 + j1.7321 + 0.7071 + j0.7071 \]  
\[ = -0.2929 - j1.0249 \]  
\[ = 1.0660e^{-j105.9481^\circ} \]  

Addition Of Multiple Sinusoids. The above calculation shows that the addition of two sinusoids of the same frequency is equivalent to the addition of complex phasors. The same is true if we are adding many sinusoids of the same frequency. This is illustrated by the following algebraic steps:

\[ f(t) = \sum_{k=1}^{n} |X_k| \cos(\omega_0 t + \angle X_k) \]  
\[ = \sum_{k=1}^{n} |X_k| \text{Re}\{e^{j(\omega_0 t + \angle X_k)}\} \]  
\[ = \sum_{k=1}^{n} \text{Re}\{|X_k| e^{j(\omega_0 t + \angle X_k)}\} \]  
\[ = \sum_{k=1}^{n} \text{Re}\{|X_k| e^{j\angle X_k} e^{j\omega_0 t}\} \]  
\[ = \text{Re}\{\sum_{k=1}^{n} |X_k| e^{j\angle X_k} e^{j\omega_0 t}\} \]
\[ \begin{align*}
\text{where} \\
X &= |X| e^{j\angle X} = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} |X_k| e^{j\angle X_k}. 
\end{align*} \]

### 2.2 Results From Trigonometry

The following relationships are useful in the study of dynamical systems.

\[
\begin{align*}
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \quad (2.70) \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \quad (2.71) \\
\tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \quad (2.72) \\
\sin(x \pm 90^\circ) &= \pm \cos(x) \quad (2.73) \\
\cos(x \pm 90^\circ) &= \mp \sin(x) \quad (2.74) \\
\tan(x \pm 90^\circ) &= -\tan(x) \quad (2.75) \\
\sin x \cos y &= \frac{1}{2} [\sin(x + y) + \sin(x - y)] \quad (2.76) \\
\cos x \cos y &= \frac{1}{2} [\cos(x + y) + \cos(x - y)] \quad (2.77) 
\end{align*}
\]
\[
\sin^2 x = \frac{1}{2} [1 - \cos 2x] \\
\cos^2 x = \frac{1}{2} [1 + \cos 2x] \\
\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x} \\
\]

\[
\sin^2 x + \cos^2 x = 1 \\
\tan^2 x + 1 = \sec^2 x \\
\cot^2 x + 1 = \csc^2 x
\]

### 2.3 Results From Calculus

The following relationships are useful in the study of dynamical systems.

\[
f'(x) \overset{\text{def}}{=} \lim_{\Delta \to 0} \left[ \frac{f(x + \Delta) - f(x)}{\Delta} \right] \\
\]

\[
\frac{d}{dx} \sin x = \cos x \\
\frac{d}{dx} \cos x = -\sin x \\
\frac{d}{dx} \tan x = \sec^2 x \\
\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \\
\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}} \\
\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \\
\frac{d}{dx} e^{ax} = ae^x \\
\frac{d}{dx} \ln x = \frac{1}{x} \\
\]

\[
\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x) \\
\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)
\]
\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{1}{g^2(x)} \left[ f'(x)g(x) - f(x)g'(x) \right]
\] (2.95)

\[
\int udv = uv - \int vdu
\] (2.96)

**Theorem 2.3.1 (Mean Value Theorem For Differentiation.)**

If \( f(\cdot) \) is differentiable on the open interval \((x_1, x_2)\) and continuous on the closed interval \([x_1, x_2]\), then there exists \( z \in (x_1, x_2) \) such that

\[
f'(z) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.
\] (2.97)

The number \( f'(z) \) is sometimes referred to as the mean derivative or average rate of change of \( f \) on the interval \([x_1, x_2]\). □

**Theorem 2.3.2 (Mean Value Theorem For Integration.)**

If \( f(\cdot) \) is continuous on a closed interval \([x_1, x_2]\), then there exists \( z \in (x_1, x_2) \) such that

\[
f(z) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(\tau) d\tau.
\] (2.98)

The number \( f(z) \) is sometimes referred to as the mean value or average value of \( f \) on the interval \([x_1, x_2]\). □

**Theorem 2.3.3 (Fundamental Theorem Of Calculus.)**

Suppose \( f \) is continuous on the closed interval \([a, b]\).

(i) If

\[
G(x) \overset{\text{def}}{=} \int_{a}^{x} f(\tau) d\tau
\] (2.99)

for all \( x \in [a, b] \), then \( G \) is the antiderivative of \( f \) on \([a, b]\); i.e.

\[
G'(x) = \frac{d}{dx} \int_{a}^{x} f(\tau) d\tau = f(x)
\] (2.100)
for all $x \in [a,b]$.

(ii) If $F$ is any antiderivative of $f$, then

$$\int_a^b f(\tau) d\tau = F(b) - F(a). \quad (2.101)$$

This theorem was discovered independently by Sir Isaac Newton (1642-1727) in England and by Gottfried Leibniz (1646-1716) in Germany. It is primarily because of this, that they are credited with the invention of calculus.

**Theorem 2.3.4 (Taylor’s Theorem.)**

Let $f$ be a function and $n$ a positive integer such that the derivative $f^{(n+1)}(x)$ exists for every $x$ in an interval $I$. If $x_0$ and $x$ are distinct numbers in $I$, then there exists a number $z \in (x_0, x)$ such that

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{f^{(n+1)}(z)}{(n + 1)!} (x - x_0)^{n+1}. \quad (2.102)$$

This theorem was discovered by the English mathematician Brook Taylor (1685-1731).

**Algorithm 2.3.1 (Newton’s Method For Solving $f(x) = 0$).**

Newton’s method for solving the equation

$$f(x) = 0 \quad (2.103)$$
is based on a sequence of tangent line approximations. The method is described by the following recursive algorithm:
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]
(2.104)

where \( n = 0, 1, 2, \ldots \). Convergence ideas are discussed in [47].
Chapter 3

Mathematical Models For Continuous Time Signals

In this section, mathematical models for typically encountered continuous time signals (functions) are presented [34], [36].

*Function Notation.* In what follows, a function $f$ will be denoted by the notation $f(\cdot)$ or simply $f$. At times the symbol $f(t)$ might be used. It must be noted, however, that strictly speaking $f(t)$ does not represent the function $f$ but rather, the value of the $f$ at $t$. Despite this, the symbol $f(t)$ is widely used to represent $f$.

3.1 A Model for Tall Thin Pulse-like Signals: Dirac Delta Distribution

In the study of systems, it is useful to use test signals which are pulse-like in nature. A model for tall thin pulse-like signals possessing unit area is now developed. Let $p_\epsilon(\cdot)$ denote the tall thin pulse defined as follows:

$$p_\epsilon(t) \overset{\text{def}}{=} \begin{cases} 
0 & -\infty < t < -\epsilon/2 \\
\frac{1}{\epsilon} & -\epsilon/2 \leq t < \epsilon/2 \\
0 & \epsilon/2 \leq t < \infty
\end{cases} \quad (3.1)$$

where $\epsilon$ represents a small positive quantity. $p_\epsilon$ defines a pulse with height $\frac{1}{\epsilon}$, width $\epsilon$, which is symmetric with respect to the origin. As $\epsilon$ approaches zero, the above pulse becomes thinner and
taller, while its area remains unity (cf. Figure 3.1). In the limit \( p_\varepsilon \) approaches a “function” which has infinite value at \( t = 0 \) and is zero elsewhere. This limit certainly does not describe a traditional function. A fundamental sifting property of \( p_\varepsilon(\cdot) \) is now observed. Let \( f(\cdot) \) denote a function which is continuous in a small \( \varepsilon \)-neighborhood of the point \( t = a \). Given this, it follows that for sufficiently small \( \varepsilon \)

\[
\int_{-\infty}^{\infty} f(\tau) p_\varepsilon(\tau - a) \, d\tau = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(\tau) \frac{1}{\varepsilon} \, d\tau \approx \frac{1}{\varepsilon} \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(a) \, d\tau = f(a)
\]

(3.2)

Moreover,

\[
\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} f(\tau) p_\varepsilon(\tau - a) \, d\tau = f(a)
\]

(3.3)

This relationship is referred to as a sifting property because it states that the pulse \( p_\varepsilon(\cdot - a) \) with center at \( t = a \), allows the value \( f(a) \) to pass through the integral - in the same manner that a wiremesh sieve (pronounced “siv”) allows certain material to pass through it. This sifting property motivates the introduction of a new “non-traditional function” - the unit Dirac delta distribution. Its definition is now given.

![Figure 3.1: Tall Thin Pulse-Like Signals](image)

**Definition 3.1.1 (Unit Dirac Delta Distribution.)**

The unit Dirac delta distribution is denoted \( \delta(\cdot) \) and defined by the following three properties:

1. **Integral Sifting Property:**

\[
\int_{-\infty}^{\infty} f(\tau) \delta(\tau - a) \, d\tau = \int_{a-}^{a+} f(\tau) \delta(\tau - a) \, d\tau \overset{\text{def}}{=} f(a)
\]

(3.4)
ii. Even Symmetry Property:

\[ \delta(t) \overset{\text{def}}{=} \delta(-t) \quad (3.5) \]

iii. Infinite Height Zero Width Property:

\[ \delta(t) \overset{\text{def}}{=} \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad (3.6) \]

The first property implies that \( \delta \) has unit area, just like the pulse \( p_\epsilon(\cdot) \). The second property implies that \( \delta \) is symmetric with respect to the vertical axis at \( t = 0 \), just like \( p_\epsilon(\cdot) \). The last property implies that \( \delta \) has infinite height and zero width, just like \( p_\epsilon(\cdot) \) as \( \epsilon \) approaches zero. The unit Dirac delta distribution is usually depicted as shown in Figure 3.2. It is important to emphasize that the unit Dirac delta distribution is not a traditional function. Despite this, it is widely referred to as the unit Dirac delta function or the unit impulse function. Sometimes, it is referred to as a generalized function [49]. Finally, it should be noted that while the unit Dirac delta distribution only makes sense underneath integrals, it is often referred to without explicit reference to integrals.

\[ \delta(t) \quad (1) \text{ – denotes unit area} \]

\[ \delta(t) \quad t \]

Figure 3.2: Unit Dirac Delta Distribution

3.2 A Model for Step-like Signals: Unit Step Function

Step-like test signals are also useful in the study of systems. A model for such signals is the unit step function. Represented by the symbol \( 1(\cdot) \), the unit step function is defined as follows:

\[ 1(t) \overset{\text{def}}{=} \begin{cases} 0 & -\infty < t < 0 \\ 1 & 0 \leq t < \infty \end{cases} \quad (3.7) \]
From the above discussion, it follows that the unit step function $1(\cdot)$ and the unit Dirac delta distribution $\delta(\cdot)$ are related as follows:

$$\int_{-\infty}^{t} \delta(\tau)d\tau = 1(t) \quad (3.8)$$

$$\frac{d}{dt}1(t) = \delta(t). \quad \quad \quad (3.9)$$

### 3.3 Exponential Signals

The exponential function

$$f(t) = e^{at}1(t) \quad \quad (3.10)$$

is widely used to model exponential signals in the study of systems. The constant $a$ is sometimes referred to as the neper frequency. For $a > 0$, the exponential grows toward $\infty$ as $t$ approaches $\infty$. As $a$ is made more positive, the rate at which the exponential grows increases (see Figure 3.4).

For $a = 0$, the function becomes the unit step function $1(\cdot)$ discussed earlier. Finally, for $a < 0$ the exponential decays toward zero as $t$ approaches $\infty$. In such a case, we say that the exponential has a time constant $\tau \overset{\text{def}}{=} \frac{1}{|a|}$ and settles to zero in approximately five time constants. The latter, widely used, convention follows because $e^{a5\tau} = e^{-5} \approx 0.0067$ is “small.” Of course, how small is “small” depends on the specific application under consideration. As $a$ is made more negative, the associated time constant decreases and the exponential decays more rapidly toward zero. It is interesting to note that the initial rate of decay (slope) is given by $-\frac{1}{\tau}$ (see Figure 3.5).
### 3.4 Complex Exponential Signals: Creating Sinusoids and Exponential Sinusoids

The utility of complex exponential signals is succinctly conveyed by Euler’s relationships. Letting $z = |z|e^{j\angle z}$ denote an arbitrary complex quantity, $\bar{z}$ its complex conjugate, $Re\{z\}$ its real part, and $Im\{z\}$ its imaginary part, Euler’s relationships may be stated as follows \[1, \text{ pg. 42}, \text{ [41]}:\]

\[
|z| \cos \angle z = Re \{ z \} = \frac{z + \bar{z}}{2} \quad |z| \sin \angle z = Im \{ z \} = \frac{z - \bar{z}}{j2}
\]  

(3.11)

Substituting

\[
z = |X|e^{\sigma_it} e^{j(\omega_it + \angle X)} 1(t) = |X|e^{j\angle X} e^{(\sigma_i + j\omega_i)t} 1(t)
\]  

(3.12)

into the above formulae, yields the following useful relationships:

\[
|X|e^{\sigma_it} \cos(\omega_it + \angle X)1(t) = Re \{ |X|e^{\sigma_it} e^{j(\omega_it + \angle X)} \} 1(t) = Re \{ |X|e^{j\angle X} e^{(\sigma_i + j\omega_i)t} \} 1(t)
\]

(3.13)

\[
|X|e^{\sigma_it} \sin(\omega_it + \angle X)1(t) = Im \{ |X|e^{\sigma_it} e^{j(\omega_it + \angle X)} \} 1(t) = Im \{ |X|e^{j\angle X} e^{(\sigma_i + j\omega_i)t} \} 1(t).
\]

(3.14)

These clearly show that exponential sinusoids (and hence sinusoids) can be created from complex exponentials. We say that the above exponential sinusoids are generated by the complex exponential

\[
X(t) = X e^{(\sigma_i + j\omega_i)t} 1(t)
\]

(3.15)

where

\[
X \overset{\text{def}}{=} |X| e^{j\angle X}
\]

(3.16)
is called the complex phasor representation of the above real exponential sinusoids. The motivation behind the latter new terminology is that once the “complex frequency” \( s_0 = \sigma_o + j\omega_o \) is noted, the complex exponential \( X(\cdot) \) and hence the original exponential sinusoids are completely determined by the complex phasor \( X \). The above exponential sinusoids are said to have

- neper frequency \( |\sigma_o| \),
- time constant \( \tau = \frac{1}{|\sigma_o|} \) (assuming \( \sigma_o < 0 \) so that the exponentials decay),
- frequency of oscillation \( \omega_o \),
- cyclic frequency \( f = \frac{\omega_o}{2\pi} \),
- period of oscillation \( T = \frac{2\pi}{\omega_o} \),
- phase shift \( -\theta \),
- time shift \( -\frac{\theta}{\omega_o} \),
- complex frequency \( s_o = \sigma_o + j\omega_o \),
- conjugate frequency \( \bar{s}_o = \sigma_o - j\omega_o \), and a
- complex phasor representation \( X \equiv |X|e^{j\angle X} \).

The neper frequency \( \sigma_o \) defines real exponentials which bound the exponential sinusoids. The exponential sinusoids grow toward \( \infty \) if and only if \( \sigma_o = Re\{s_o\} > 0 \); i.e. \( s_o \) lies in the right
half complex plane (see Figure 3.6). The exponential sinusoids decay to zero if and only if $\sigma_o = Re\{s_o\} < 0$; i.e. $s_o$ lies in the left half complex plane (see Figure 3.7).

**Figure 3.6: Growing Exponential Sinusoid**

**Figure 3.7: Decaying Exponential Sinusoid**

In what follows, it will be shown that exponentials (real and complex) are particularly important in the study of linear systems described by linear ordinary differential equations with constant coefficients.
Chapter 4

An Introduction To Laplace Transforms

Many dynamical systems may be modeled or approximated by linear ordinary differential equations with constant coefficients. Given this, an indispensable tool for analyzing such systems is the unilateral Laplace transform [7], [12], [14], [15], [19], [28], [32], [34], [39], [43]. A definition is now provided.

4.1 The Unilateral Laplace Transform: An Introduction

Definition 4.1.1 (Unilateral Laplace Transform.) The unilateral Laplace transform of a function \( f(\cdot) \) is denoted \( F(s) = (\mathcal{L}f)(s) \) and defined as follows:

\[
F(s) = (\mathcal{L}f)(s) \overset{\text{def}}{=} \int_{0}^{\infty} f(\tau)e^{-st}d\tau.
\]  

(4.1)

The purpose of the minus sign on the lower limit is to ensure that Dirac delta distributions at the origin are captured. The set of complex numbers \( s \) for which the Laplace integral converges (i.e. makes sense, is finite) is called the region of convergence (ROC) of \( F(\cdot) \) [32].

A list of elementary Laplace transform pairs is contained in Table 4.1. A comprehensive list of Laplace transform properties is contained in Table 4.2.
<table>
<thead>
<tr>
<th>t-Description</th>
<th>f(t)</th>
<th>F(s)</th>
<th>s-Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>(a_1 f_1(t) + a_2 f_2(t))</td>
<td>(a_1 F_1(s) + a_2 F_2(s))</td>
<td>Linearity</td>
</tr>
<tr>
<td>Exponentiation</td>
<td>(e^{at} f(t))</td>
<td>(F(s-a))</td>
<td>Shift</td>
</tr>
<tr>
<td>Multiplication</td>
<td>(ct f(t))</td>
<td>((-1)^n \frac{d^n}{ds^n} F(s))</td>
<td>Differentiation</td>
</tr>
<tr>
<td>Scaling</td>
<td>(f(\frac{1}{a} t))</td>
<td>(a F(as))</td>
<td>Scaling</td>
</tr>
<tr>
<td>Shift (Delay)</td>
<td>(f(t - \Delta)1(t - \Delta))</td>
<td>(e^{-\Delta s} F(s); \Delta \geq 0)</td>
<td>Exponentiation</td>
</tr>
<tr>
<td>Differentiation</td>
<td>((s) f(t))</td>
<td>(s^n F(s) - \sum_{k=1}^{n-1} \frac{1}{(n-k)!} f(0^-))</td>
<td>Multiplication</td>
</tr>
<tr>
<td>Integration</td>
<td>(\int_0^t f(\tau) d\tau)</td>
<td>(\frac{F(s)}{s})</td>
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</tr>
<tr>
<td>Convolution</td>
<td>((f_1 * f_2)(t) \triangleq \int_0^t f_1(t - \tau) f_2(\tau) d\tau)</td>
<td>(F_1(s) F_2(s))</td>
<td>Function Multiplication</td>
</tr>
<tr>
<td>Function Multiplication</td>
<td>(f_1(t) f_2(t))</td>
<td>(\frac{1}{2\pi j} \int_{-\infty}^{+\infty} F_1(s-p) F_2(p) dp)</td>
<td>Convolution</td>
</tr>
</tbody>
</table>

Table 4.2: Laplace Transform Properties

### 4.2 Elementary Unilateral Laplace Transforms

**Example 4.2.1 (Dirac Delta Distribution.)** Show that

\[
\delta(t) \overset{L}{\rightarrow} 1. \quad (4.2)
\]

**Example 4.2.2 (Unit Step Function.)** Show that

\[
1(t) \overset{L}{\rightarrow} \frac{1}{s}, \quad \text{Re } s > 0. \quad (4.3)
\]
Solution:

Letting \( f(t) = 1(t) \), it follows that

\[
F(s) = \int_{0^-}^{\infty} e^{-st} \, dt = \left. \frac{e^{-st}}{-s} \right|_0^{\infty} = \frac{e^{-st}}{-s} \bigg|_0^{\infty}.
\]

(4.4)

Letting \( s = \sigma + j\omega \) in the above exponential yields

\[
F(s) = \frac{e^{-(\sigma+j\omega)t}}{s} \bigg|_0^{\infty} = \frac{e^{-(\sigma+j\omega)t}}{s} \bigg|_0^{\infty} = \frac{1}{s} \left[ 1 - e^{-\sigma t} e^{j\omega t} \bigg|_{t=\infty} \right].
\]

(4.5)

From this it follows that the right hand side makes sense (i.e. is finite) if and only if \( \text{Re} s = \sigma > 0 \).

It thus follows that

\[
F(s) = \frac{1}{s}
\]

(4.6)

where \( \text{Re} s > 0 \) is the region of convergence of \( F(\cdot) \); i.e. the set of \( s \) values in the complex plane for which the integral

\[
F(s) = \int_{0^-}^{\infty} e^{-st} \, dt
\]

(4.7)

makes sense is given by

\[
\text{ROC}_F = \{ s \in \mathbb{C} \mid \text{Re} s > 0 \}.
\]

(4.8)

For any other values of \( s \), the integral makes no sense. Try, for example, the values \( s = -1, 1 + j1, j1 \). For these values, the integral makes no sense.

Example 4.2.3 (Exponential Function.) Show that

\[
e^{at}1(t) \xleftrightarrow{L} \frac{1}{s-a}, \quad \text{Re} s > \text{Re} a
\]

(4.9)

where \( a \in \mathbb{C} \) is an arbitrary complex number.

Solution:

Letting \( f(t) = e^{at}1(t) \), it follows that

\[
F(s) = \int_{0^-}^{\infty} e^{at} e^{-st} \, dt = \int_{0^-}^{\infty} e^{-(s-a)t} \, dt = \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^{\infty} = \frac{e^{-(s-a)t}}{-(s-a)} \bigg|_0^{\infty}.
\]

(4.10)
Letting $s = \sigma + j\omega$ in the above exponential yields

$$F(s) = \frac{e^{-(\sigma + j\omega - \alpha)\tau}}{s - a} \bigg|_0^\infty = \frac{e^{-(\sigma + j\omega - \alpha)\tau}}{s - a} \bigg|_0^\infty = \frac{1}{s - a} \left[1 - e^{-(\sigma - \alpha)\tau} e^{j\omega\tau} \right]_{\tau \to \infty}.$$  \hspace{1cm} (4.11)

From this it follows that the right hand side makes sense (i.e. is finite) if and only if $\text{Re } s = \sigma > \text{Re } a$. It thus follows that

$$F(s) = \frac{1}{s - a}$$  \hspace{1cm} (4.12)

where $\text{Re } s > \text{Re } a$ is the region of convergence of $F(\cdot)$; i.e. the set of $s$ values in the complex plane for which the integral

$$F(s) = \int_0^\infty e^{-(s - \alpha)\tau} d\tau$$  \hspace{1cm} (4.13)

makes sense is given by

$$\text{ROC}_F = \{s \in \mathbb{C} \mid \text{Re } s > \text{Re } a\}.$$  \hspace{1cm} (4.14)

For any other values of $s$, the integral makes no sense. [1]

In the above example, the parameter $a$ may be real, purely imaginary, or complex.

**Example 4.2.4 (Exponentiation and Frequency Shifting Property.)** Show that

$$e^{at} f(t) \overset{\mathcal{L}}{\longrightarrow} F(s - a)$$  \hspace{1cm} (4.15)

where $a \in \mathbb{C}$ is an arbitrary complex number. [1]

**Example 4.2.5 (Exponentiation and Frequency Shifting Property.)** Show that

$$e^{\sigma t} \sin(\omega_0 t + \theta) \overset{\mathcal{L}}{\longrightarrow} \frac{\sin \theta(s - \sigma_0) + \omega_0 \cos \theta}{s^2 - 2\sigma_0 s + \sigma_0^2 + \omega^2} \quad \text{Re } s > \sigma_0$$  \hspace{1cm} (4.16)
Example 4.2.6 (Scaling Property.) Show that

$$f\left(\frac{t}{a}\right) \overset{\mathcal{L}}{\longleftrightarrow} aF(as) \quad (4.17)$$

for any real constant $a$.

Example 4.2.7 (Time Multiplication and Frequency Differentiation Property.) Show that

$$tf(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}F(s). \quad (4.18)$$

Example 4.2.8 (Powers of $t$.) Show that

$$\frac{t^n}{n!}1(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^{n+1}} \quad (4.19)$$

for $n = 0, 1, 2, \ldots$.

Example 4.2.9 (Time Differentiation Property.) Show that

$$\frac{d}{dt}f(t) \overset{\mathcal{L}}{\longleftrightarrow} sF(s) - f(0^-) \quad (4.20)$$

$$\frac{d^2}{dt^2}f(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2F(s) - sf(0^-) - \dot{f}(0^-) \quad (4.21)$$

$$\frac{d^3}{dt^3}f(t) \overset{\mathcal{L}}{\longleftrightarrow} s^3F(s) - s^2f(0^-) - sf(0^-) - \ddot{f}(0^-). \quad (4.22)$$

Example 4.2.10 (Higher Order Deltas.) Show that

$$\delta^{(n)}(t) \overset{\mathcal{L}}{\longleftrightarrow} s^n \quad (4.23)$$
Example 4.2.11 (Time Delay Property.) Show that

\[ f(t - \Delta)1(t - \Delta) \overset{\mathcal{L}}{\rightarrow} e^{-s\Delta} F(s) \tag{4.24} \]

for any \( \Delta \geq 0 \).

Example 4.2.12 (Time Convolution-Frequency Multiplication Property.) Show that

\[ \int_0^t f(t - \tau)g(\tau)d\tau \overset{\mathcal{L}}{\rightarrow} F(s)G(s). \tag{4.25} \]

Example 4.2.13 (Integration Property.) Show that

\[ \int_0^t f(\tau)d\tau \overset{\mathcal{L}}{\rightarrow} \frac{F(s)}{s}. \tag{4.26} \]

Example 4.2.14 (Initial Value Theorem.) Suppose that \( F(\cdot) \) is a real-rational function. Show that

\[ f(0^+) = \lim_{s \to \infty} sF(s). \tag{4.27} \]

Example 4.2.15 (Final Value Theorem.) Suppose that \( F(\cdot) \) is a real-rational function. Show that if the limit

\[ f(\infty) \overset{\text{def}}{=} \lim_{t \to \infty} f(t) \tag{4.28} \]
exists, then

\[ f(\infty) = \lim_{s \to 0} sF(s). \]  

(4.29)

## 4.3 Applications

The following example is used to illustrate how Laplace transform techniques are applied to analyze systems described by linear ordinary differential equations with constant coefficients. It is also used to initiate the introduction of many fundamental system concepts.

### Example 4.3.1 (Analysis Of A First Order System.)

Consider a single-input single-output (SISO) system with input \( u(\cdot) \) and output \( y(\cdot) \), related by the following first order linear ordinary differential equation with constant coefficients:

\[ \dot{y}(t) = -ay(t) + bu(t) \quad y(0^-) = y_0 \quad t > 0 \]  

(4.30)

where \( a, b > 0 \). The purpose of this example is to illustrate elementary Laplace transform techniques and to initiate the introduction of fundamental system concepts.

1. Input-Output Modeling. To provide motivation, it is useful to note that the above input-output model may be used to approximate the input-output behavior of many physical systems. These might include

   - an RC series circuit with \( y \) and \( u \) representing voltages,
   - an autonomous vehicle with \( y \) representing vehicle speed and \( u \) representing an applied force,
   - a mechanical shaft with \( y \) representing the angular speed of the shaft and \( u \) representing an applied torque,
• a semiconductor diffusion process with \( y \) representing substrate temperature and \( u \) representing an applied power.

II. System Order. Since our system is modeled by a first order differential equation, we say that we have a first order system.

III. Analysis Via Laplace Transforms. Let \( Y(\cdot) = (\mathcal{L}y)(\cdot) \) and \( U(\cdot) = (\mathcal{L}u)(\cdot) \). Taking the Laplace transform of each side of equation (4.30) then yields

\[
sY(s) - y_0 = -aY(s) + bU(s)
\]

(4.31)

or

\[
Y(s) = \frac{y_0}{s + a} + \left[ \frac{b}{s + a} \right] U(s).
\]

(4.32)

This shows that \( Y(\cdot) \) consists of two terms. The first term is due to initial conditions. The second term is due to the system input (or forcing function) \( u(\cdot) \).

IV. Characteristic Equation, System Poles, And Natural Modes. Both terms in equation (4.32) have the common denominator

\[
\Phi(s) = s + a.
\]

(4.33)

This polynomial is called the characteristic equation or characteristic polynomial of the system. The roots of this equation are called the system poles. Given this, it follows that our system has a single pole at

\[
s = -a
\]

(4.34)
in the left half complex \( s \)-plane (since \( a > 0 \)). From Laplace transforms, we know that this pole corresponds to a decaying exponential

\[
e^{-at} u(t)
\]

(4.35)
in the time domain. We thus say that our system possesses a natural exponential decaying mode.

V. Linear System. Equation (4.32) shows that $Y(\cdot)$ is linear in \([ y_0 \quad U(\cdot) ]\). That is, if \([ y_{0i} \quad U_i(\cdot) ]^T\) produces $Y_i(\cdot)$ \((i = 1, 2)\), then

$$
\begin{bmatrix}
y_0 \\
U(\cdot)
\end{bmatrix} = a_1 \begin{bmatrix}
y_{01} \\
U_1(\cdot)
\end{bmatrix} + a_2 \begin{bmatrix}
y_{02} \\
U_2(\cdot)
\end{bmatrix} \implies Y(\cdot) = a_1 Y_1(\cdot) + a_2 Y_2(\cdot).
$$

(4.36)

Since the inverse Laplace operator $\mathcal{L}^{-1}$ is also a linear operator, it follows that $y(\cdot)$ is linear in \([ y_0 \quad u(\cdot) ]\); i.e. if \([ y_{0i} \quad u_i(\cdot) ]^T\) produces $y_i(\cdot)$ \((i = 1, 2)\), then

$$
\begin{bmatrix}
y_0 \\
u(\cdot)
\end{bmatrix} = a_1 \begin{bmatrix}
y_{01} \\
u_1(\cdot)
\end{bmatrix} + a_2 \begin{bmatrix}
y_{02} \\
u_2(\cdot)
\end{bmatrix} \implies y(\cdot) = a_1 y_1(\cdot) + a_2 y_2(\cdot).
$$

(4.37)

Because of this, we say the system under consideration is a linear system.

VI. Zero Input Response. The first term in equation (4.32) is identified as the Laplace transform of the zero input response

$$
y_{zir}(t) = y_0 e^{-at} 1(t);
$$

(4.38)

i.e. $Y_{zir}(s) \overset{\text{def}}{=} (\mathcal{L} y_{zir})(s) = \frac{y_0}{s+a}$. The function $y_{zir}(\cdot)$ represents the natural response of the system to an initial condition $y_0$ with no input \((u = 0)\); i.e. it is a solution to the homogeneous equation

$$
\dot{y}(t) = -ay(t) \quad y(0^-) = y_0 \quad t > 0.
$$

(4.39)

The term "homogeneous equation" comes from differential equations. The natural response exhibits the natural modes of the system. Our system, as mentioned above, has a natural exponential decaying mode.

VII. Equilibrium And Exponential Stability. From the original system differential equation (equation (4.30)), one notes that when $u = 0$, and hence $\dot{y} = -ay$, the system has a natural equilibrium of $y = 0$. That is, when left alone with $y_0 = u = 0$, the system remains at $y = 0$. From equation (4.38), we see that if the system is given an offset $y_0$ then it naturally returns back (exponentially) to its
equilibrium \( y = 0 \). Because of this, we say that the system, or more precisely the equilibrium \( y = 0 \) is exponentially stable.

VIII. System Transfer Function. From equation (4.32), it follows that

\[
H(s) \overset{\text{def}}{=} \frac{(Ly)(s)}{(Lu)(s)}_{\text{zero initial conditions}}
\]

\[
= \frac{b}{s + a}.
\]

The region of convergence of \( H(\cdot) \) is given by

\[
\text{ROC}_H = \{ s \in \mathbb{C} \mid \text{Re } s > -a \}.
\] (4.42)

This region defines a half plane which lies to the left of the system pole at \( s = -a \). This zero initial condition ratio is called the transfer function of the system. It is important to note that the system transfer function completely determines the system input-output linear differential equation model and vice versa. That is, knowledge of one implies knowledge of the other. This follows from the very important zero initial condition transform pair

\[
\mathcal{L} \left \{ f(t) \right \} \overset{\mathcal{L}}{\rightarrow} \mathcal{S}(s).
\] (4.43)

IX. System Impulse Response. Now suppose that \( y_0 = 0 \) and that the input is a unit Dirac delta distribution; i.e. \( u(t) = \delta(t) \). Under these conditions, \( Y(s) = H(s) \) and hence \( y(t) = h(t) \overset{\text{def}}{=} (\mathcal{L}^{-1}H)(t) \). Because of this, we call

\[
h(t) \overset{\text{def}}{=} (\mathcal{L}^{-1}H)(t) = be^{-at}1(t)
\] (4.44)

the impulse response of the system. As with the system transfer function \( H(\cdot) \), the system impulse response \( h(\cdot) \) completely specifies the system differential equation. Moreover, knowledge of any one of the following:

- system ordinary differential equation,
- system transfer function \( H(\cdot) \),
• system impulse response $h(\cdot)$,

implies knowledge of the other two.

X. General Solution And Convolution. From equation (4.32), it follows that if $y_o$ and $U(\cdot)$ are known, then partial fraction expansion inversion methods can be used to determine $y(\cdot)$. While this powerful technique is available, it is also very useful to note that

$$y(t) = y_{cir}(t) + (h * u)(t)$$

$$= y_{cir}(t) + \int_0^t h(t-\tau)u(\tau)d\tau \quad t > 0$$

(4.45) (4.46)

where $(h * u)(\cdot)$ denotes the convolution of the system impulse response $h(\cdot)$ with the system input $u(\cdot)$. This shows that convolution occurs naturally for systems described by linear ordinary differential equations with constant coefficients. Equation (4.46) gives us another method for determining $y(\cdot)$ given $y_o$ and $u(\cdot)$ (and knowledge of $h(\cdot)$). While the first term is called the zero input response, the second term is called the forced response. The second term is also a particular solution to the system differential equation (assuming zero initial conditions). The term “particular solution” comes from differential equations.

XI. Casual System. From equation (4.46), it follows that the current output $y(t)$ depends on current and past values of the input $u(\cdot)$. This implies that the system cannot produce an output $y(\cdot)$ until an input $u(\cdot)$ is applied. Because of this, we say that the system under consideration is a causal system.

XII. Bounded-Input Bounded-Output (BIBO) Stability. Suppose that the system is initially inert (i.e. $y_o = 0$). This implies that

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t u(t-\tau)h(\tau)d\tau.$$  

(4.47)

Suppose that a bounded input $u(\cdot)$ is applied to the system. Specifically, suppose that

$$|u(t)| \leq M$$

(4.48)
for all $t$. Given this, it follows that

$$
|y(t)| \leq \int_0^t |u(t - \tau)h(\tau)|d\tau
$$

(4.49)

$$
\leq \int_0^t M|h(\tau)|d\tau
$$

(4.50)

$$
\leq M\int_0^t |h(\tau)|d\tau
$$

(4.51)

$$
\leq M\int_0^\infty |h(\tau)|d\tau
$$

(4.52)

$$
\leq M\int_0^\infty be^{-\alpha\tau}d\tau
$$

(4.53)

$$
\leq M\frac{b}{\alpha}.
$$

(4.54)

This implies that a bounded-input $u(\cdot)$ results in a bounded-output $y(\cdot)$. We thus say that the system under consideration is bounded-input bounded-output (BIBO) stable.

XIII. Time Invariance. Suppose that $y_o = 0$. It then follows that $Y(s) = H(s)U(s)$. Suppose $u_1(\cdot)$ produces $y_1(\cdot)$. What is the output $y(\cdot)$ when a delayed input $u(t) = u_1(t - \Delta)$ ($\Delta \geq 0$) is applied to the system? This input results in

$$
Y(s) = H(s)U_1(s) = H(s)U(s)e^{-s\Delta} = Y_1(s)e^{-s\Delta}.
$$

(4.55)

and hence $y(t) = y_1(t - \Delta)$; i.e. shifted inputs result in shifted outputs. Because of this, we say that our system is a time invariant system. Fundamentally, our system is time invariant because the coefficients in the differential equation are constant. Because they are constant, the “shape” of the output doesn’t depend on when the input is applied.

XIV. Analysis At $t = t_o$. Suppose that the input $u(\cdot)$ is applied at some time $t = t_o \in \mathcal{R}$. That is, suppose that we are interested in

$$
\dot{y}(t) = -ay(t) + bu(t) \quad \quad y(t_o) = y_o \quad \quad t > t_o.
$$

(4.56)

In such a case, we have

$$
y(t) = e^{-a(t-t_o)}y_o + \int_{t_o}^t h(t - \tau)u(\tau)d\tau \quad \quad t > t_o.
$$

(4.57)
XV. Concept Of State, State Variables, State Equation. The state of the system at time $t_o$ is defined to be the minimum amount of information required to compute the future evolution of the system output $y(\cdot)$, given knowledge of the input $u(\cdot)$. With this definition, it follows from equation (4.57) that the state of the system at $t = t_o$ is precisely the initial condition $y_0$. (One cannot, for example, compute the speed of a vehicle given knowledge of the applied force if the initial speed of the vehicle is not known!) Since the state of the system at time $t_o$ is $y(t_o^-) = y_0$ for any $t_o$, we call $y(\cdot)$ a state variable. If we let $x \equiv y$ denote the state of the system, then equation (4.57) may be rewritten as follows:

$$
\dot{x}(t) = -ax(t) + bu(t) \quad \text{ (4.58)}
$$

$$
y(t) = x(t). \quad \text{ (4.59)}
$$

Since the first equation involves the state $x$, the equation is called a state equation. Since the second equation relates the output $y$ to the system state $x$ (and the system input $u$), the equation is called an output equation. The two equations constitute a state space representation for the system.

XVI. Zero State Response. Just as $y_{zir}(\cdot)$ denotes the zero input response, the above terminology leads one to call the second term (forced response, $(h \ast u)(\cdot)$) in equation (4.57), the zero state response (ZSR); i.e. it is the response of the system to the input $u(\cdot)$ assuming zero initial conditions ($y_0 = 0$, zero initial state).

XVII. Step Response. The response to a step input of size $k$ is found by setting $U(s) = \frac{k}{s}$ and using partial fraction expansion methods as follows:

$$
Y(s) = H(s)U(s) = \left[ \frac{b}{s + a} \right] \frac{k}{s} = H(0)k \left[ \frac{1}{s} - \frac{1}{s + a} \right] = \frac{b}{a}k \left[ \frac{1}{s} - \frac{1}{s + a} \right]. \quad \text{ (4.60)}
$$

From this it follows that the step response is given by

$$
y(t) = H(0)k \left[ 1 - e^{-at} \right] 1(t) = \frac{b}{a}k \left[ 1 - e^{-at} \right] 1(t). \quad \text{ (4.61)}
$$
The exponential portion of equation (4.61) is due to the system’s natural exponential mode. This portion is called the transient portion of the response or the transient response. The constant portion of equation (4.61) is due to the system input. This portion is called the steady state portion of the response or the steady state response.

The step response may also be computed using

\[ y(t) = \int_0^t u(t - \tau)h(\tau)d\tau = k \int_0^t h(\tau)d\tau. \] (4.62)

Given this, we note further that the unit impulse response \( h(\cdot) \) and the unit step response \( s(\cdot) \) \((k = 1)\) are related by the following formulae:

\[ s(t) = \int_0^t h(\tau)d\tau \] (4.63)

\[ h(t) = \frac{d}{dt}s(t). \] (4.64)

Figure 4.1 illustrates how the step response of the system depends on the parameter \( a \) with \( b = a \).

As \( a \) increases, the response time decreases.

![Step Response of First Order System: \( H(s) = \frac{a}{s+a}, \ a = [0.25, 0.5, 0.75, 1] \)](image)

**Figure 4.1: Step Response Of A First Order System**
XVIII. Steady State Response To Step Input. From equation (4.61), it follows that as \( t \to \infty \), the output \( y(\cdot) \) approaches the steady state value:

\[
y_{ss} = \lim_{t \to \infty} y(t)H(0)k = \frac{b}{a}
\]

where

\[
H(s)\big|_{s=j0} = \frac{b}{a}
\]

is called the dc gain of the system.

XIX. Response To A Sinusoid. Now suppose that an input \( u(t) = |X|\cos(\omega_0 t + \angle X) \) is applied to the system. Setting

\[
U(s) = |X| \left[ \cos \angle X s - \omega_0 \sin \angle X \right] \frac{s^2 + \omega_0^2}{s^2 + \omega_0^2}
\]

yields

\[
Y(s) = \left[ \frac{b}{s + a} \right] |X| \left[ \cos \angle X s - \omega_0 \sin \angle X \right] \frac{s + a}{s + a + s - j\omega_0} + *
\]

where * denotes the conjugate of the preceding term. From this it follows that

\[
y(t) = \left[ Ae^{-at} + 2|B|\cos(\omega_0 t + \angle B) \right] 1(t)
\]

where

\[
A = \lim_{s \to -a} (s + a)Y(s) = -|X|b \frac{(a \cos \angle X + \omega_0 \sin \angle X)}{a^2 + \omega_0^2}
\]

\[
B = \lim_{s \to j\omega_0} (s - j\omega_0)Y(s) = \frac{H(j\omega_0)}{2} |X|e^{j\angle X}.
\]

The first term in equation (4.69) is due to the system’s natural decaying exponential mode. This term is called the transient response since it goes away as \( t \to \infty \). The second term in equation (4.69) is due to the sinusoidal forcing function \( u(\cdot) \) and is called the steady state response.
XX. Steady State Response To A Sinusoid. From equation (4.69), one obtains the steady state response $y_{ss}$ to a sinusoid $u(\cdot)$ as follows:

$$ u(t) = |X| \cos(\omega_0 t + \angle X) \quad y_{ss} = |H(j\omega_0)||X| \cos(\omega_0 t + \angle X + \angle H(j\omega_0)) $$

$$ = \text{Re} \left\{ |X| e^{j\angle X} e^{j\omega_0 t} \right\} \quad \text{produc}es \quad = \text{Re} \left\{ H(j\omega_0)|X| e^{j\angle X} e^{j\omega_0 t} \right\} $$

$$ = \text{Re} \left\{ X e^{j\omega_0 t} \right\} \quad = \text{Re} \left\{ H(j\omega_0)X e^{j\omega_0 t} \right\} $$

(4.72)

where

$$ X \overset{\text{def}}{=} |X| e^{j\angle X} $$

(4.73)

is called the phasor representation of the input signal $u(\cdot)$. Equation (4.72) implies that in the steady state, the input cosine with frequency $\omega_0$, amplitude $|X|$, and angle $\angle X$, comes out of the system as a cosine signal with the same frequency $\omega_0$, with new amplitude $|H(j\omega_0)||X|$, and new angle $\angle X + \angle H(j\omega_0)$. That is, the amplitude of the input is modified by a multiplicative factor $|H(j\omega_0)|$ and the angle is modified by an additive factor $\angle H(j\omega_0)$.

XXI. Continuous Time Fourier Transform. The unilateral continuous time Fourier transform of a function $f(\cdot)$ is denoted $(\mathcal{F}f)(j\cdot)$ and defined as follows:

$$ (\mathcal{F}f)(j\omega) \overset{\text{def}}{=} \int_{0}^{\infty} f(\tau) e^{-j\omega \tau} d\tau. $$

(4.74)

In our example, the region of convergence for $H(\cdot) \overset{\text{def}}{=} (\mathcal{L}h)(\cdot)$ includes the imaginary axis. Given this, it follows that for the system under consideration $H(j\cdot)$ is the Fourier transform of $h(\cdot)$. In general, however,

$$ H(j\omega) \overset{\text{def}}{=} (\mathcal{L}h)(s) \mid_{s=j\omega} \neq (\mathcal{F}h)(j\omega) $$

(4.75)

because $s = j\omega$ may not lie within the region of convergence of $H(\cdot) \overset{\text{def}}{=} (\mathcal{L}h)(\cdot)$. An example illustrating this point is easily obtained [34, pg. 307], [36]:

$$ h(t) = u(t) \quad H(s) \overset{\text{def}}{=} (\mathcal{L}h)(s) = \frac{1}{s} \quad H(j\omega) = \frac{1}{j\omega} \neq (\mathcal{F}h)(j\omega) = \frac{1}{j\omega + \pi \delta(\omega)}. $$

(4.76)
In general,

\[(\mathcal{L}h)(s)|_{s=j\omega} = (\mathcal{F}h)(j\omega)\]  \hfill (4.77)

if and only if the \(s = j\omega\) lies within the region of convergence of \((\mathcal{L}h)(s)\). The above statements also apply when working with bilateral Laplace and Fourier transforms:

\[(\mathcal{L}h)(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \]  \hfill (4.78)

\[(\mathcal{F}h)(j\omega) \overset{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau. \]  \hfill (4.79)

XXII. Frequency Response. The function \(H(j\omega) \overset{\text{def}}{=} (\mathcal{L}h)(s)|_{s=j\omega}\) is called the frequency response of the system. The function \(|H(j\omega)|\) is called the magnitude response of the system. Its relevance is succinctly conveyed by equation (4.72). For our system, the magnitude response function is given as follows:

\[|H(j\omega)| = \frac{b}{\sqrt{\omega^2 + a^2}}. \]  \hfill (4.80)

The function \(\angle H(j\omega)\) is called the phase response of the system. Its relevance is also succinctly conveyed by equation (4.72). For our system, the magnitude response function is given as follows:

\[\angle H(j\omega) = -\tan^{-1}\frac{\omega}{a}. \]  \hfill (4.81)

Figure 4.2 illustrates how the frequency response of the system depends on the parameter \(a\) with \(b = a\). In this figure, \(20\log_{10}|H(j\omega)|\) is plotted on a linear vertical scale versus \(\omega\) in radians/second on a horizontal logarithmic (base 10) scale. We say that \(|H(j\omega)|\) is plotted in unit of decibels (dB) versus \(\omega\) on a semilog grid. As \(a\) is increased, the system magnitude response is seen to shift towards the right. In the same figure, \(\angle H(j\omega)\) is plotted in degrees on a linear vertical scale versus \(\omega\) in radians/second on a horizontal logarithmic (base 10) scale. As \(a\) is increased, the angle response is seen to shift towards the right.
XXIII. Bandwidth. The bandwidth of a system is a term which is aimed at quantifying the ability or inability of a system to amplify rapidly varying signals. The most widely used definition for bandwidth is as follows. The bandwidth of a system is defined to be that frequency $BW$ at which

$$|H(j\omega)| = H(0) \frac{1}{\sqrt{2}}.$$  \hspace{1cm} (4.82)

For our system, we have

$$BW = a.$$  \hspace{1cm} (4.83)

Given this, Figure 4.2 shows that as $a$ is increased the system bandwidth increases. This explains why the step response looks more and more like a step as $a$ is increased.

The following example illustrates how elementary transform techniques may be used to address linear systems described by linear ordinary differential equations with constant coefficients.

Example 4.3.2 \textbf{(5$^\text{th}$ Order Linear Ordinary Differential Equation.)}
Suppose that a system with input \( u(\cdot) \) and output \( y(\cdot) \) is described by the following 5th order linear ordinary differential equation with constant coefficients

\[
y^{(5)}(t) + a_4 y^{(4)}(t) + a_3 y^{(3)}(t) + a_2 y^{(2)}(t) + a_1 y(t) + a_0 y(t) = u(t)
\]

where

\[
s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = s(s+1)^2(s^2 + 4s + 13).
\]

In this example, we illustrate elementary transform techniques and we introduce fundamental system concepts. We proceed in several steps.

I. System Transfer Function. Assuming zero initial conditions, we begin by determining the ratio

\[
H(s) = \frac{(\mathcal{L}y)(s)}{(\mathcal{L}u)(s)}_{\text{zero initial conditions}}.
\]

This ratio is called the transfer function of the system. Taking Laplace transforms of both sides of equation (4.84) yields the real-rational s-domain function

\[
H(s) = \frac{1}{s(s+1)^2(s^2 + 4s + 13)}.
\]

II. Sinusoidal Response. The goal now is to determine \( y(\cdot) \) when \( u(t) = |X| \sin(\omega_o t + \angle X) \) \( 1(t) \).

This requires that we take the Laplace transform of equation (4.84) and use the fact that

\[
U(s) = \frac{-\sin \angle X s + \omega_o \cos \angle X}{s^2 + \omega_o^2}
\]

to obtain

\[
Y(s) = H(s)U(s) = \frac{-\sin \angle X s + \omega_o \cos \angle X}{s(s+1)^2(s^2 + 4s + 13)(s^2 + \omega_o^2)}.
\]

To determine the inverse transform \( y(\cdot) \) it is essential to note that this \( Y(\cdot) \) has poles at \( s = 0, -1, -1, -2 \pm j3, \pm j\omega_o \). These poles are critical for determining \( y \). Next, expand \( Y \) in a partial fraction expansion [32] as follows:

\[
Y(s) = \frac{A}{s} + \frac{B}{(s+1)^2} + \frac{C}{s + 1} + \frac{D}{s + 2 - j3} + * + \frac{E}{s - j\omega_o} + *
\]
where $A, B, C, D, E$ are constants to be determined below and $\ast$ denotes the complex conjugate of the previous term in the expansion. In performing the above partial fraction expansion, we have used the fact that the sum is a rational function with real coefficients and hence the complex quantities must occur in complex conjugate pairs. Taking the inverse Laplace transform of each term yields:

$$y(t) = [A + Bt e^{-t} + Ce^{-t} + De^{-(2+j3)t} + \ast + Ee^{j\omega_t} + \ast] 1(t) \quad (4.91)$$

Substituting $D = |D|e^{j\angle D}$ and $E = |E|e^{j\angle E}$ into the above expression yields:

$$y(t) = [A + Bt e^{-t} + Ce^{-t} + |D|e^{-(2t + j\angle D)} + \ast + |E|e^{j(\omega_t + \angle E)} + \ast] 1(t) \quad (4.92)$$

Simple application of Euler’s formula then yields:

$$y(t) = [A + Bt e^{-t} + Ce^{-t} + 2Re\{|D|e^{-(2t + j\angle D)}\} + 2Re\{|E|e^{j(\omega_t + \angle E)}\}] 1(t) \quad (4.93)$$

or

$$y(t) = [A + Bt e^{-t} + Ce^{-t} + 2|D|e^{-(2t \cos(3t + \angle D)) + 2|E| \cos(\omega_t t + \angle E)}] 1(t) \quad (4.94)$$

It now remains to determine the coefficients $A, B, C, D, E$. These can be determined using the following standard partial fraction expansion formulae [32, pp. 35-44]:

$$A = \lim_{s \to 0} s \Re \{Y(s)\} = \frac{-\sin \angle Xs + \omega_0 \cos \angle X}{(s + 1)^2(s^2 + 4s + 13)(s^2 + \omega_0^2)} |_{s=0} \quad (4.95)$$

$$B = \lim_{s \to -1} (s + 1)^2 \Re \{Y(s)\} = \frac{-\sin \angle Xs + \omega_0 \cos \angle X}{s(s^2 + 4s + 13)(s^2 + \omega_0^2)} |_{s=-1} \quad (4.96)$$

$$C = \lim_{s \to -2+j3} \frac{d}{ds}[(s + 1)^2 \Re \{Y(s)\}] = \frac{d}{ds}[\frac{\sin \angle Xs + \omega_0 \cos \angle X}{s(s^2 + 4s + 13)(s^2 + \omega_0^2)}] |_{s=-2+j3} \quad (4.97)$$

$$D = \lim_{s \to -j\omega_0} (s + 2 - j3) \Re \{Y(s)\} = \frac{-\sin \angle Xs + \omega_0 \cos \angle X}{s(s + 1)^2(s + 2 + j3)(s + j\omega_0)} |_{s=-j\omega_0} \quad (4.98)$$

$$E = \lim_{s \to j\omega_0} (s - j4) \Re \{Y(s)\} = \frac{-\sin \angle Xs + \omega_0 \cos \angle X}{s(s + 1)^2(s^2 + 4s + 13)(s + j\omega_0)} |_{s=j\omega_0} \quad (4.99)$$

III. Sinusoidal Steady State Analysis. It is useful to note that for large $t$, $y$ may be approximated as follows:

$$y(t) \approx [A + 2|E| \cos(\omega_0 t + \angle E)] 1(t) \quad (4.100)$$
The righthand side is called the steady state component of \( y \) and is denoted

\[
y_{ss} = \left[ A + 2|E| \cos(\omega_0 t + \angle E) \right] 1(t) \tag{4.101}
\]

Specifically, it is useful to note that the steady state component due to the input sinusoid \( u(\cdot) \) is also a sinusoid. Specifically, it can be shown that

\[
E = \lim_{s \to j\omega_0} (s - j\omega_0)Y(s) = \frac{-\sin \angle Xs + \omega_0 \cos \angle X}{s(s + 1)^2(s^2 + 4s + 13)(s + j\omega_0)} |_{s = j\omega_0} = \frac{H(j\omega_0)}{2} e^{\angle X \text{ deg}} \tag{4.102}
\]

This, however implies that

\[
\begin{align*}
u(t) &= |X| \sin(\omega_0 t + \angle X) \\
&= \text{Im} \left\{ |X| e^{j\angle X} e^{j\omega_0 t} \right\} \quad \text{produces} \quad y_{ss_{\text{sinusoid}}} = |H(j\omega_0)||X| \cos(\omega_0 t + \angle X + \angle H(j\omega_0) - 90 \text{ deg}) \\
&= \text{Im} \left\{ H(j\omega_0)|X| e^{j\angle X} e^{j\omega_0 t} \right\} \\
&= \text{Im} \left\{ H(j\omega_0)|X| e^{j\omega_0 t} \right\}
\end{align*}
\]

where

\[
X \overset{\text{def}}{=} |X| e^{j\angle X} \tag{4.104}
\]

is called the phasor representation of the input signal \( u(\cdot) \). Equation (4.103) implies that in the steady state, the input sine wave with frequency \( \omega_0 \), amplitude \( |X| \), and angle \( \angle X \), results in a steady state sinusoidal component with the same frequency \( \omega_0 \), with new amplitude \( |H(j\omega_0)||X| \), and new angle \( \angle X + \angle H(j\omega_0) \). That is, the amplitude of the input is modified by a multiplicative factor \( |H(j\omega_0)| \) and the angle is modified by an additive factor \( \angle H(j\omega_0) \).

V. Frequency Response. The function \( H(j\omega) \) is called the frequency response of the system. The function \( |H(j\omega)| \) is called the magnitude response of the system. Its relevance is succinctly conveyed by equation (4.103). The function \( \angle H(j\omega) \) is called the phase or angle response of the system. Its relevance is also succinctly conveyed by equation (4.103).
Chapter 5

SISO Input-Output Models: LTI Systems

5.1 An Introduction To Linear Systems: An Illustrative Example

The following simple RLC series circuit example will be used to motivate, introduce, and illustrate many of the linear system concepts to follow.

Example 5.1.1 (Series RLC Circuit - A Single-Input Single-Output (SISO) System.)

Consider the series RLC circuit shown in Figure 5.1. The circuit, or system, input (forcing function) is the voltage source $u$. The circuit, or system, output is the voltage measurement $y$. In this circuit, the resistor, inductor, and capacitor values are denoted $R$, $L$, and $C$ respectively. It is assumed that they are nonnegative and constant. The symbol $i_L$ is used to represent the current flowing through the inductor and $y$ represents the voltage across the capacitor.

![Figure 5.1: Single-Input Single-Output (SISO) Series RLC Circuit](image)

This simple single-input single-output (SISO) example will be used to motivate/illustrate the following concepts/techniques: input-output model, system order, analysis via Laplace transforms,
natural response, stability, characteristic equation, characteristic roots, poles, damping factor, system transfer function, general RLC circuit impedance methods, impulse response, general solution, convolution, linearity, causality, step response, overshoot, undershoot, time invariance, linear time invariance (LTI), bounded-input bounded-output stability, sinusoidal steady state analysis, zeros, frequency response, magnitude response, phase response, Fourier transform, descriptor representations, state, state variables, and state space representations. As such, the example provides an excellent “fuel-rich” vehicle for embarking on our study of linear systems. Moreover, it will be shown that ALL of the principles and concepts to be illustrated using this very simple example extend effortlessly to arbitrarily complex systems described by linear ordinary differential equations with constant coefficients.

5.2 LTI System Concepts

The following example is used to motivate and introduce various important system concepts.

Example 5.2.1 (Series RLC Circuit Input-Output Model Analysis.)

Consider the series RLC circuit in Figure 5.1.

Input-Output Linear Differential Equation Model. One can use Kirchhoff’s voltage and current laws [23] to relate the output $y(t)$ to the input $u(t)$. Doing so, yields the following second order linear ordinary differential equation with constant coefficients:

\[
\ddot{y}(t) + \frac{R}{L} \dot{y}(t) + \frac{1}{LC} y(t) = \frac{1}{LC} u(t). \tag{5.1}
\]

This equation is subject to the two initial conditions: $y(0^-) = y_0$, $\dot{y}(0^-) = \dot{y}_0$. Equation (5.1) relates the output $y(t)$ to the input $u(t)$ for $t > 0$. As such, we say that it defines a linear differential equation input-output model for the system. Because the system is described by a second order differential equation, we say that the system is a second order system. It should be noted
that we have obtained a second order model because we have assumed the presence of two energy storage elements, namely $L$ and $C$. It should also be noted that modeling other (perhaps difficult to measure) parasitic inductances and capacitances would increase the system order.

Analysis Via Laplace Transforms. It is now shown how Laplace transform techniques may be used to analyze systems described by linear ordinary differential equations with constant coefficients such as equation (5.1). The opportunity is taken to introduce fundamental terminology as well.

Let $Y(\cdot)$ and $U(\cdot)$ denote the Laplace transforms of the output $y(\cdot)$ and the input $u(\cdot)$, respectively. Now apply the transform properties in Table (4.2) (time differentiation property) to the differential equation (5.1). Doing so yields:

$$[ s^2 Y(s) - sy_0 - \dot{y}_0 ] + \frac{R}{L} [ sY(s) - y_0 ] + \frac{1}{LC} [ Y(s) ] = \frac{1}{LC} [ U(s) ].$$  \hspace{1cm} (5.2)

Solving for $Y(s)$ yields:

$$Y(s) = \left[ \frac{sy_0 + \dot{y}_0 + \frac{R}{L} y_0}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \right] + \left[ \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \right] U(s).$$  \hspace{1cm} (5.3)

Natural (Zero Input) Response. At this point, it is useful to note that the first term is due to initial conditions. We therefore identify

$$Y_{zir}(s) = \frac{sy_0 + \dot{y}_0 + \frac{R}{L} y_0}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{(s + \frac{R}{L})y_0 + \dot{y}_0}{s^2 + \frac{R}{L}s + \frac{1}{LC}},$$  \hspace{1cm} (5.4)

as the Laplace transform of the zero input response $y_{zir}(\cdot)$; i.e. the response to initial conditions with no input ($u = 0$). The zero input response captures the natural tendencies or natural modes of the system. A detailed zero input response partial fraction analysis for our RLC circuit will be given subsequently.

System Transfer Function. It is useful to write equation (5.3) as

$$Y(s) = Y_{zir}(s) + H(s)U(s)$$  \hspace{1cm} (5.5)
where

\[
H(s) \equiv \frac{(\mathcal{L}y)(s)}{(\mathcal{L}u)(s)} \big|_{\text{zero initial conditions}}
\]

\[
= \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}
\]

is called the system transfer function.

System Output and Convolution. Given values for the initial conditions \((y_o, \dot{y}_o)\) and the input \(u(\cdot)\), one can (in principle) always use partial fraction techniques to determine the system output \(y(\cdot)\) - assuming, of course, that all relevant Laplace transforms and inverse transforms are computable.

Despite the availability of powerful partial fraction techniques, it is instructive to note that the system output \(y(\cdot)\) is related to system initial conditions \((y_o, \dot{y}_o)\) and the system input \(u(\cdot)\) as follows:

\[
y(t) = y_{\text{dir}}(t) + (Tu)(t)
\]

\[
y(t) = y_{\text{dir}}(t) + (h \ast u)(t)
\]

\[
y(t) = y_{\text{dir}}(t) + \int_0^t h(t - \tau)u(\tau)d\tau
\]

where

\[
(Tu)(t) \equiv (h \ast u)(t)
\]

\[
= \int_0^t h(t - \tau)u(\tau)d\tau.
\]

denotes the system’s convolution operator, \((h \ast u)(\cdot)\) represents the convolution of \(h(\cdot)\) with \(u(\cdot)\), and

\[
h(t) \equiv (\mathcal{L}^{-1}H)(t).
\]

System Impulse Response. Suppose that all initial conditions are set to zero \((y_o = \dot{y}_o = 0)\) and that a unit Dirac delta distribution is applied to the system; i.e. \(u(t) = \delta(t)\). In such a case, it follows that \(Y(s) = H(s)\) and hence \(y(t) = h(t)\). Because of this, we call

\[
h(t) \equiv (\mathcal{L}^{-1}H)(t)
\]
the impulse response of the system. It should be noted that \( h(t) = 0 \) for all \( t < 0 \). The impulse response for our RLC circuit may be computed using partial fraction techniques.

Causal System. Many systems do not exhibit an output until some input is applied to the system. Such systems are said to be causal. More precisely, our RLC circuit is a causal system because the current output \( y(t) \) depends on the present and/or previous inputs \( u(\tau), \tau \in [0,t] \). This follows from the integral limits in equation (5.10). The lower integral limit follows from the assumption that the input is applied at \( t = 0 \). The upper limit of \( t \) follows from the fact that the impulse response \( h(\cdot) \) is zero for all \( t < 0 \). This fact implies that \( h(t - \tau) = 0 \) for \( t < \tau \). This, in turn, forces the upper limit in equation (5.10) to be \( t \) rather than \( \infty \).

Linear System. From equation (5.3), it follows that \( Y(\cdot) \) is linear in \((y_0, \dot{y}_0, U(\cdot))\). Since the inverse Laplace operator \( \mathcal{L}^{-1} \) is also a linear operator, it follows that the output \( y(\cdot) \) is linear in \((y_0, \dot{y}_0, u(\cdot))\); i.e.

The output \( y(\cdot) \) is linear in the initial conditions and the input \( u(\cdot) \).

That is, if \([y_0, \dot{y}_0, u_1(\cdot)]^T \) produces an output \( y_1(\cdot) \) and \([y_0, \dot{y}_0, u_2(\cdot)]^T \) produces an output \( y_2(\cdot) \), then the linear combination \([y_0, \dot{y}_0, u(\cdot)]^T = a_1[y_0, \dot{y}_0, u_1(\cdot)]^T + a_2[y_0, \dot{y}_0, u_2(\cdot)]^T \) produces the output \( y(\cdot) = a_1 y_1(\cdot) + a_2 y_2(\cdot) \). For this reason, we say that our RLC circuit is a linear system.

Time Invariant System. For many systems, the response to a given input \( u \) is “time invariant,” i.e. shifting the input results in a shifted output. To show that this is true for our RLC circuit, define the the delay operator

\[
(S_\Delta w)(t) \overset{\text{def}}{=} w(t - \Delta).
\]  

(5.15)

Now consider the following (zero initial condition) calculation:

\[
(TS_\Delta u)(t) = \int_0^t h(t - \tau)u(\tau - \Delta)d\tau
\]  

(5.16)
\[ = \int_{t-\Delta}^{t} h(t - \Delta - \alpha)u(\alpha)d\alpha \quad \alpha \overset{\text{def}}{=} \tau - \Delta \quad d\alpha = d\tau \quad (5.17) \]
\[ = \int_{0}^{t-\Delta} h(t - \Delta - \alpha)u(\alpha)d\alpha \quad (5.18) \]
\[ = \int_{0}^{t-\Delta} h(t - \Delta - \alpha)u(\alpha)d\alpha \quad (5.19) \]
\[ = y(t - \Delta) \quad (5.20) \]
\[ = (S_{\Delta}Tu)(t) \quad (5.21) \]

In this calculation, we have assumed that the input has been applied at \( t = 0 \). It shows that our RLC circuit is a time invariant system.

Linear Time Invariant (LTI) System. Because our RLC circuit is linear and time invariant, we say that it is a linear time invariant (LTI) system. We will study LTI systems in great detail.

Observations: Structure of Output. At this point, some general observations are in order. \( Y(\cdot) \) and its inverse \( y(\cdot) \) consists of two components (cf. equation (5.10)):

1. Natural Response. The first term (\( Y_{\text{zir}}(\cdot) \) or \( y_{\text{zir}}(\cdot) \)) is due to initial conditions and hence captures the natural response of the system; i.e. the response of the system to initial conditions with no input applied \((u = 0)\). The denominator of \( Y_{\text{zir}}(\cdot) \) (or that of \( H(\cdot) \)) is called the characteristic equation of the system. Its roots are called the system poles. These poles correspond to time domain signals which represent the natural tendencies or natural modes of the system.

2. Forced Response. The second term is due to the system input (forcing function) \( u(\cdot) \). More precisely, it is due to the system input “exciting” the system transfer function \( H(\cdot) \) and hence the system’s natural modes. The idea being conveyed here is best understood in the s-domain. Both \( U(\cdot) \) and \( H(\cdot) \) have poles. These poles correspond to time domain functions which help make up the second term (forced response). The second term thus has a contribution due to the system transfer function \( H(\cdot) \) and a contribution due to the input \( U(\cdot) \).
**SISO LTI Systems Described By Linear Differential Equation Models.** The significance of the above observations lies in the fact that they apply to arbitrarily complex LTI systems. In what follows, we will examine LTI systems described by linear ordinary differential equations with constant coefficients as follows:

\[
\sum_{k=0}^{n_1} a_k^{(k)} y(t) = \sum_{k=0}^{n_2} b_k^{(k)} u(t) \quad (5.22)
\]

\[
Y(s) = Y_{zr}(s) + H(s)U(s) \quad (5.23)
\]

\[
Y_{zr}(s) = \frac{\sum_{k=0}^{n_1} a_k \left[ s^{k-1} y_0 + s^{k-2} y_0 + \cdots + s y_0 + y_0 \right]}{\sum_{k=0}^{n_1} a_k s_k} \quad (5.24)
\]

\[
H(s) \overset{\text{def}}{=} \left( \frac{Ly}{Lu} \right)(s)_{\text{zero initial conditions}} = \frac{\sum_{k=0}^{n_2} b_k s^k}{\sum_{k=0}^{n_1} a_k s_k} \quad (5.25)
\]

\[
h(t) \overset{\text{def}}{=} (L^{-1}H)(t) \quad (5.26)
\]

\[
y(t) = y_{zr}(t) + (h \ast u)(t) \quad (5.27)
\]

\[
y_{zr}(t) + \int_0^t h(t - \tau)u(\tau)d\tau \quad (5.28)
\]

where \(y^{(k)}(0^+) = \left( \frac{d}{dt} \right)^k y_0\) denote initial time derivatives. In equation (5.24), it has been assumed that all initial conditions associated with the input \(u\) are zero; i.e. \(\{u_0 = 0\}_k\). If the latter initial input derivatives are not zero, then another (similar) term must be added in equation (5.24).

**System Order.** For systems described by linear differential equation models (cf. equation (5.22)), the order of the system is defined to be the order of the highest derivative appearing in the differential equation model; i.e. \(n \overset{\text{def}}{=} \max(n_1, n_2)\).

**SISO LTI Systems Described By Linear Integral Models.** In what follows, we will also consider LTI (possibly noncausal) systems described by *linear integral models* as follows:

\[
y(t) = (Tu)(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} h(t - \tau)u(\tau)d\tau. \quad (5.29)
\]
For such (possibly noncausal) systems, the transfer function is defined by a bilateral Laplace transform as follows:

\[ H(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} u(\tau)d\tau. \] (5.30)

### 5.3 Natural Response, Stability, Characteristic Equation, And Poles

To deeply understand any system, it is important to understand the natural tendencies or natural modes of the system. Two physical examples are given to illustrate these and related ideas (e.g. equilibrium, stability, etc.).

**Example 5.3.1 (An Upright Broomstick.)**

Consider an upright broomstick. When left completely alone, it remains upright - assuming a perfectly flat, albeit infinitesimally thin, bottom. We say that the upright position is an equilibrium for the broomstick. When the broomstick is given a small initial offset (initial condition), its natural tendency is to fall (assuming no human input is applied). One can use Newton's laws to show that its angle with respect to the vertical grows exponentially as

\[ e^{\sqrt{g}\tau} \] (5.31)

where \( g \) is the acceleration due to gravity and \( l \) is the length of the broomstick. This shows that the broomstick possesses a natural growing exponential mode. Moreover, since the broomstick does not return to its vertical equilibrium and actually moves further away from it, we say that the vertical equilibrium is an unstable equilibrium. We also say that the broomstick possesses a natural mode which is unstable. From Laplace transforms we know that this instability has associated with it a right half plane (unstable) pole at

\[ s = \sqrt{\frac{g}{l}} \] (5.32)
since

\[ \mathcal{L}\{e^{\sqrt{\frac{k}{l}} t}\} = \frac{1}{s - \sqrt{\frac{k}{l}}} \]  \hspace{1cm} (5.33)

\[ \n\]

Example 5.3.2 (Swinging Pendulum Within Grandfather Clock.)

Now consider the pendulum which swings within a grandfather clock. When left alone, the pendulum naturally remains downward. We say that the downward position is an equilibrium for the pendulum. When given a small initial offset (initial condition), the pendulum’s natural tendency is to return to its downward equilibrium position. Newton’s laws can be used to show that the angle of the pendulum with respect to the vertical decays like an exponential sinusoid

\[ e^{-\frac{k}{l} t} \cos \left( \sqrt{\frac{g}{l} - \left(\frac{k}{2}\right)^2} t + \theta \right) \]  \hspace{1cm} (5.34)

where \( k \) represents a friction coefficient, \( g \) the acceleration due to gravity, and \( l \) the length of the pendulum. This shows that the pendulum possesses a natural decaying exponential sinusoidal mode. Moreover, since the pendulum returns to its downward equilibrium exponentially, we say that the downward equilibrium is an exponentially stable equilibrium. We also say that the pendulum possesses a natural mode which is exponentially stable. From Laplace transforms we know that this mode has associated with it a left half plane (stable) pole at

\[ s = -\frac{k}{2} \pm j \sqrt{\frac{g}{l} - \left(\frac{k}{2}\right)^2}. \]  \hspace{1cm} (5.35)

The above examples suggest that in order to examine the natural tendencies or natural modes of a system, it is essential to study the natural (unforced) response (zero input response) of the system; i.e. the response due to initial conditions with no input \( u = 0 \). Doing so also tells us about
the stability of the system.

**Stability.** For systems described by linear differential equations (with coefficients that are possibly time varying), we say that the system is

- **unstable** if initial conditions produce time responses which do not remain bounded,
- **stable or marginally stable** if initial conditions produce bounded time responses,
- **asymptotically stable** if initial conditions decay to zero,
- **exponentially stable** if initial conditions decay to zero exponentially.

**Stability For Finite-Dimensional LTI Systems.** For systems described by linear ordinary differential equations with constant coefficients (and hence real-rational transfer functions), asymptotic and exponential stability are equivalent. Such systems are

\[
\text{exponentially stable if and only if all of the system poles}
\]

\[
\text{lie in the right half complex s-plane; i.e. have negative real parts.}
\]

Other notions of stability will be presented subsequently.

**Example 5.3.3 (Natural Modes Of Standard Second Order System.)**

Consider the RLC series circuit in Figure 5.1. An input-output model for this circuit is given by equation (5.1).

Characteristic Equation, Poles. To understand the natural modes of this system, and its stability, one begins with an examination of its characteristic equation:

\[
\Phi(s) \overset{\text{def}}{=} s^2 + \frac{R}{L} s + \frac{1}{LC}.
\]  

(5.36)

It’s roots are called the characteristic roots of the system, the poles of the system, or simply the system poles. These poles completely determine the natural tendencies or modes of the system.
Standard Second Order Form. At this point, it is convenient to write the characteristic equation \( \Phi(\cdot) \) as follows:

\[
\Phi(s) = s^2 + 2\zeta\omega_n s + \omega_n^2
\]

where

\[
\omega_n \overset{\text{def}}{=} \frac{1}{\sqrt{LC}}
\]

is called the undamped natural frequency and

\[
\zeta \overset{\text{def}}{=} \frac{R}{2}\sqrt{\frac{C}{L}}
\]

is called the damping factor. When written in this manner, we say that the characteristic equation is in standard second order form.

System Poles, Natural Response Dependence On Damping Factor \( \zeta \). With this notation, the poles of the system are given by

\[
s_1 = -\zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1} \quad s_2 = -\zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1}
\]

for \( \zeta \geq 1 \) and

\[
s_1 = -\zeta\omega_n + j\omega_n \sqrt{1 - \zeta^2} \quad s_2 = -\zeta\omega_n - j\omega_n \sqrt{1 - \zeta^2}
\]

for \( \zeta \in [0,1) \). The natural (unforced) response of this system to initial conditions depends greatly on the nature of these poles. Given this, it is useful to examine how the system poles depend on the damping factor \( \zeta \) (pronounced “zeta”). Four cases arise.

Case 1: Undamped (\( \zeta = 0 \)). When \( \zeta = 0 \), the system is said to be undamped. In such a case the system poles are distinct, purely imaginary, complex conjugates of one another, and correspond to
complex exponentials $e^{\pm j\omega_n t}$ or a sinusoid. In such a case, the natural response of the system is an undamped oscillation. More precisely,

$$y_{nat}(t) = [2|A| \cos(\omega_n t + \angle A)] 1(t) \tag{5.42}$$

where

$$A = \left. \frac{sy_0 + \dot{y}_o + \frac{R}{L}y_o}{s + j\omega_n} \right|_{s = j\omega_n}. \tag{5.43}$$

In such a case, the system is said to be marginally stable.

Case 2: Underdamped ($0 < \zeta < 1$). When $0 < \zeta < 1$, the system is said to be underdamped. In such a case the system poles are distinct, complex, complex conjugates of one another, and correspond to decaying (exponentially stable) complex exponentials $e^{(-\zeta \omega_n \pm j\omega_n \sqrt{1-\zeta^2}) t}$ or an exponential sinusoid. More precisely,

$$y_{nat}(t) = [2|A|e^{-\zeta \omega_n t} \cos(\omega_n \sqrt{1-\zeta^2} t + \angle A)] 1(t) \tag{5.44}$$

where

$$A = \left. \frac{sy_0 + \dot{y}_o + \frac{R}{L}y_o}{s + \zeta \omega_n + j\omega_n \sqrt{1-\zeta^2}} \right|_{s = -\zeta \omega_n + j\omega_n \sqrt{1-\zeta^2}}. \tag{5.45}$$

For this case

- $\sigma = -\zeta \omega_n$ is the neper frequency,
- $\tau = \frac{1}{\zeta \omega_n}$ is the time constant,
- $\omega_d \overset{\text{def}}{=} \omega_n \sqrt{1-\zeta^2}$ is the damped natural frequency,
- $T = \frac{2\pi}{\omega_d}$ is the period of oscillation,
- $s = \sigma + j\omega_d$ is the complex frequency, and
- $s = \sigma - j\omega_d$ is the conjugate complex frequency.

As $\zeta$ is made slightly positive, the system poles move into the left half $s$-plane, the system becomes underdamped, and the system’s natural response becomes less oscillatory than for the $\zeta = 0$ (undamped) case. For small $\zeta$, the natural response of the system is still very oscillatory. As $\zeta$ is
increased, the response becomes less oscillatory.

Case 3: Critically Damped ($\zeta = 1$). When $\zeta = 1$, the system is said to be critically damped. In such a case the system poles are repeated (both at $s = -\omega_n$), real, and correspond to decaying (exponentially stable) exponentials $e^{-\omega_n t}$, $te^{-\omega_n t}$. More precisely,

$$y_{dr}(t) = [ tA e^{-\omega_n t} + Be^{-\omega_n t} ] 1(t) \quad (5.46)$$

where

$$A = -\omega_n y_0 + \frac{R}{L} \dot{y}_0 \quad B = y_0. \quad (5.47)$$

For this $\zeta$, oscillations are no longer present in the system’s natural response and the system time constant is $\tau = \frac{1}{\omega_n}$.

Case 4: Overdamped ($\zeta > 1$). When $\zeta > 1$, the system is said to be overdamped. In such a case, the system poles are distinct, real, and correspond to decaying (exponentially stable) exponentials $e^{-(\zeta \omega_n + \sqrt{\zeta^2 - 1}) t}$, $e^{-(\zeta \omega_n - \sqrt{\zeta^2 - 1}) t}$. More precisely,

$$y_{dr}(t) = [ Ae^{s_1 t} + Be^{s_2 t} ] 1(t) \quad (5.48)$$

where

$$A = \frac{s_1 y_0 + \dot{y}_0 + \frac{R}{L} \dot{y}_0}{s_1 - s_2} \quad B = \frac{s_2 y_0 + \dot{y}_0 + \frac{R}{L} \dot{y}_0}{s_2 - s_1}. \quad (5.49)$$

As $\zeta$ is increased above unity, the system poles split - one moving to the right, the other to the left. Because $\tau_1 = \frac{1}{|s_1|} > \tau_2 = \frac{1}{|s_2|}$, it follows that $s_1$ is the slow pole and $s_2$ is the fast pole. Generally, left half plane poles near the origin are slower than poles which are further to the left. As $\zeta$ is increased significantly, the slow pole $s_1$ dominates the natural response. For $\zeta > 0$, the system is exponentially stable.
If one defines the critically damped resistance as follows:

\[ R_c \overset{\text{def}}{=} 2\sqrt{\frac{L}{C}} \]  

(5.50)

then our RLC circuit is undamped/underdamped for \( R \in (0, R_c) \), critically damped for \( R = R_c \), and overdamped for \( R \in (R_c, \infty) \).

The above discussion shows how the system poles and natural response depend on the damping factor \( \zeta \). The dependence of the poles on \( \zeta \) may be visualized as indicated in the root locus plot shown in Figure 5.2. It is interesting to note that for \( 0 \leq \zeta \leq 1 \), the system poles lie on a circle of radius \( \omega_n \) with center at \( s = 0 \).

Figure 5.2: Pole Dependence on Damping Factor: Standard Second Order Polynomial

Natural Response: General Finite-Dimensional System Modal and Stability Analysis. Consider a system whose input-output model yields the following general \( n^{\text{th}} \) order characteristic equation:

\[
\Phi(s) \overset{\text{def}}{=} \sum_{k=0}^{n} a_k s^k \quad (a_n \overset{\text{def}}{=} 1) 
\]

(5.51)

\[
= K_1 \prod_{k=1}^{K_1} (s - p_k) \prod_{k=1}^{K_2} (s^2 + 2\zeta_k \omega_n \omega_{n_k}, s + \omega_{n_k}^2) 
\]

(5.52)
where \( n = K_1 + 2K_2 \), \( a_k, p_k \in \mathbb{R} \), \( \zeta_k \in (-1, 1) \), and \( \omega_{n_k} \in (0, \infty) \). The roots of \( \Phi(\cdot) \) are precisely the poles of the system.

Associated with the first product are \( K_1 \) real poles \( \{ p_k \}_{k=1}^{K_1} \) and hence \( K_1 \) real exponential modes

\[
\{ A_k e^{p_k t} \}_{k=1}^{K_1}.
\]

Associated with the second product are \( 2K_2 \) complex poles \( \{ -\zeta_k \omega_{n_k} \pm j \omega_{n_k} \sqrt{1 - \zeta_k^2} \}_{k=1}^{K_2} \). These are associated with \( 2K_2 \) complex exponential modes which occur in complex conjugate pairs or \( K_2 \) real exponential sinusoidal modes

\[
\{ |B_k| e^{-\zeta_k \omega_{n_k} t} \cos(\omega_{n_k} \sqrt{1 - \zeta_k^2} t + \angle B_k) \}_{k=1}^{K_2}.
\]

Given the above, it follows that the above general system is

- **exponentially stable** if and only if all of the poles lie in the left half s-plane;
- **marginally stable** if and only if all of the poles lie in the left half s-plane with the possible exception of unrepeated poles on the imaginary axis.
- **unstable** if and only if any one of its poles lies in the right half plane and/or if there is a repeated pole on the imaginary axis.

### 5.4 System Transfer Function

In this section, we consider transfer functions for SISO LTI systems.

**Standard Second Order Transfer Function.** The transfer function

\[
H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

is said to be in \textit{standard second order form}. This standard form appears everywhere in the study of more complex SISO LTI systems described by linear ordinary differential equations with constant

Consider the RLC series circuit in Figure 5.1. Assuming zero initial conditions, the ratio $Y(s)/U(s)$ is given by

$$H(s) \overset{\text{def}}{=} \left( \frac{\mathcal{L}(y)(s)}{\mathcal{L}(u)(s)} \right)_{\text{zero initial conditions}}$$

$$= \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.$$  \hspace{2cm} (5.56)

This ratio is called the system transfer function. This transfer function may be written in standard second order form as follows:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$  \hspace{2cm} (5.58)

where $\omega_n \overset{\text{def}}{=} \frac{1}{\sqrt{LC}} \in (0, \infty)$ and $\zeta \overset{\text{def}}{=} \frac{R}{2} \sqrt{\frac{C}{L}} \in [0, \infty)$. \hfill \blacksquare

General Real-Rational Transfer Function. Consider a SISO LTI system with real-rational transfer function

$$H(s) \overset{\text{def}}{=} \left( \frac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))} \right)_{\text{zero initial conditions}} = \frac{\sum_{k=0}^{n_2} b_ks^k}{\sum_{k=0}^{n_1} a_ks^k} \quad (a_{n_1} \overset{\text{def}}{=} 1)$$  \hspace{2cm} (5.59)

When written in this form, we will say that it is written in standard polynomial form. It is often useful to write this transfer function in the following standard pole-zero-gain form:

$$H(s) = g \prod_{k=1}^{\text{NRP}} (s - p_k) \prod_{k=1}^{\text{NZP}} (s - z_k) \prod_{k=1}^{\text{NCPP}} (s^2 + 2\zeta_k \omega_{n_k} s + \omega_{n_k}^2)$$

where $a_k, b_k \in \mathbb{R}$,

- $\text{NRP}$ is the number of finite real poles and the $\{p_k\}$ are real poles ($p_k \in \mathbb{R}$),

- $\text{NCPP}$ is the number of finite complex pole pairs and the $\{-\zeta_{p_k} \omega_{n_{p_k}} \pm j \omega_{n_{p_k}} \sqrt{1 - \zeta_{p_k}^2}\}$ are complex poles ($\zeta_{p_k} \in (-1, 1)$, $\omega_{n_{p_k}} \in (0, \infty)$).
• NRZ is the number of finite real zeros and the \( z_k \) are real zeros \( (z_k \in \mathbb{R}) \),

• NCZP is the number of finite complex zero pairs and the \( \{ -\zeta_{n_{z_k}} \omega_{n_{z_k}} \pm j\omega_{n_{z_k}} \sqrt{1 - \zeta_{n_{z_k}}^2} \} \) are complex zeros \( (\zeta_{n_{z_k}} \in (-1, 1), \omega_{n_{z_k}} \in (0, \infty) ) \), and

• the system’s leading gain \( g \in \mathbb{R} \) is given by

\[
g \overset{\text{def}}{=} \lim_{s \to \infty} s^{n_1 - n_2} H(s) = b_{n_2}. \tag{5.61}
\]

If \( n_2 \leq n_1 \), then the above system is said to have \( n_1 - n_2 \) zeros at \( \infty \). If \( n_2 > n_1 \), then the above system is said to have \( n_2 - n_1 \) poles at \( \infty \). With this method of “accounting” (counting poles and zeros at \( \infty \)) the number of poles is always equal to the number of system zeros.

Because the above SISO LTI systems possess a finite number of poles, they are referred to as finite-dimensional systems. They are sometimes referred to as lumped parameter systems.

It should be noted that for SISO LTI systems with a real-rational transfer function, the system transfer function is readily obtainable from the system input-output differential equation and vice versa. This is always the case - for any SISO LTI system described by linear ordinary differential equations with constant coefficients. Specifically, this “equivalence” between time domain differential equations with constant coefficients and s-domain transfer functions follows from the following (zero initial condition) transform pair:

\[
\dot{y}(t) \longleftrightarrow sY(s). \tag{5.62}
\]

This pair suggests that the Laplace transform variable \( s \) is fundamentally a differential operator.

Example: Analysis Of General RLC Circuits Via Impedance Methods.

For general RLC circuits, transfer functions and hence differential equations, are readily obtainable if one exploits the circuit concept of impedance. This concept generalizes the concept of
resistance. The impedance of a resistor $R$ is

$$Z_R \overset{\text{def}}{=} \frac{V_R(s)}{I_R(s)} \overset{\text{zero initial conditions}}{=} R.$$  \hfill (5.63)

The impedance of an inductor $L$ is

$$Z_L \overset{\text{def}}{=} \frac{V_L(s)}{I_L(s)} \overset{\text{zero initial}}{=} sL.$$  \hfill (5.64)

The impedance of a capacitor $C$ is

$$Z_C \overset{\text{def}}{=} \frac{V_C(s)}{I_C(s)} \overset{\text{zero initial}}{=} \frac{1}{sC}.$$  \hfill (5.65)

With this impedance concept, arbitrary RLC circuits may be analyzed in the $s$-domain by merely treating the impedances as if they were resistances.

Consider the RLC series circuit in Figure 5.1. Given the above impedance ideas, one can use voltage division (in the $s$-domain) to readily obtain the system transfer function from the input $u(\cdot)$ to the output voltage $y(\cdot)$ across the capacitor. Doing so yields the following:

$$\frac{Y(s)}{U(s)} \overset{\text{zero initial conditions}}{=} \frac{1}{sC} \frac{1}{R + sL + \frac{1}{sC}}$$

$$= \frac{1}{s^2 + \frac{RI}{L}s + \frac{1}{LC}}.$$ \hfill (5.66)

$$= \frac{\frac{1}{LC}}{s^2 + \frac{RI}{L}s + \frac{1}{LC}}.$$ \hfill (5.67)

Analogous ideas exist for other dynamical systems.

**Irrational Transfer Functions.** Not all LTI systems are associated with real-rational transfer functions. Transfer functions which are not real-rational are said to be *irrational*. Such systems are often referred to as *infinite-dimensional systems* or *distributed parameter systems* [10].

**Example 5.4.1 (A Delay System.)**
Consider the SISO LTI system described by the linear ordinary (delay) differential equation with constant coefficients:

\[ \dot{y}(t) = -y(t) + u(t - \Delta) \]  \hspace{1cm} (5.68)

where \( \Delta > 0 \) represents a time delay. The system transfer function is given by

\[ H(s) \overset{\text{def}}{=} \frac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))}\bigg|_{\text{zero initial conditions}} = \frac{e^{-s\Delta}}{s + 1}. \]

(5.69)

This is an irrational transfer function.

Example 5.4.2 (An Irrational Transfer Function.)

Consider the LTI system described by the following linear integral convolution equation:

\[ y(t) = (Tu)(t) \overset{\text{def}}{=} \int_{-\infty}^{t} h(t - \tau)u(\tau) \] \hspace{1cm} (5.70)

with impulse response \( h(\cdot) \) given by the zero\(^{th}\) order Bessel function

\[ h(t) = J_0(t) \overset{\text{def}}{=} \left[ 1 - \frac{t^2}{2^2} + \frac{t^4}{2^24^2} - \frac{t^6}{2^24^26^2} + \cdots \right] 1(t). \]

(5.71)

It can be shown that the system transfer function is given by \([43, \text{pg. 23}]\)

\[ H(s) \overset{\text{def}}{=} \frac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))}\bigg|_{\text{zero initial conditions}} = \frac{1}{\sqrt{s^2 + 1}}. \]

(5.72)

This is an irrational transfer function.

5.5 Impulse Response

It has been shown that for an LTI system with transfer function \( H(\cdot) \), the impulse response \( h(\cdot) \) is given by

\[ h(t) = (\mathcal{L}^{-1}H)(t). \]

(5.73)

The impulse response of an LTI system is important for various reasons.
• The impulse response \( h(\cdot) \) determines the transfer function \( H(\cdot) \);

• The impulse response \( h(\cdot) \) determines the systems’ input-output model (e.g. linear ordinary differential equation with constant coefficients, convolution integral, etc.).

The impulse response \( h(\cdot) \) of a system corresponding to a real-rational transfer function \( H(\cdot) \) can be readily found using standard partial fraction expansion techniques [32].

Example 5.5.1 (RLC Series Circuit Impulse Response.)

Consider the RLC circuit in Figure 5.1. It should be noted that the qualitative statements which were made for the circuit’s zero input response in an earlier example, also apply to the circuit’s impulse response. This is because the roots of the characteristic polynomial are also poles of the system transfer function.

5.6 General Solution And Convolution

It has been shown that the general solution for an LTI system described by a linear ordinary differential equation with constant coefficients, takes the form:

\[
y(t) = y_{zir}(t) + (h \ast u)(t) \tag{5.74}
\]

\[
y(t) = y_{zir}(t) + \int_0^t h(t - \tau)u(\tau)d\tau \tag{5.75}
\]

where \( y_{zir}(\cdot) \) denotes the system’s zero input (or natural) response and \( (h \ast u)(\cdot) \) represents the convolution of \( h(\cdot) \) with \( u(\cdot) \).

A substitution of variables shows that convolution is commutative; i.e.

\[
(h \ast u)(t) = \int_0^t h(t - \tau)u(\tau)d\tau = \int_0^t u(t - \tau)h(\tau)d\tau = (u \ast h)(t). \tag{5.76}
\]

From this, it follows that

\[
y(t) = y_{zir}(t) + (u \ast h)(t) \tag{5.77}
\]
\[ y_{y_{\text{in}}} (t) + \int_0^t u(t - \tau)h(\tau)d\tau. \] (5.78)

In principle, once \( h(\cdot) \) is known, equation (5.75) or (5.78) may be used to determine the output \( y(\cdot) \) to any input \( u(\cdot) \) - given the initial conditions. \( y_{y_{\text{in}}} (\cdot) \) may be found using partial fraction Laplace inversion techniques.

**Convolution Integral Evaluation Methods.** It has been shown how convolution integrals may be evaluated using Laplace transform techniques. It must be noted that *graphical convolution methods* [34] may be used to evaluate convolution integrals. If the integrands in equations (5.75) and (5.78) are too complex, then the convolution integrals may have to be evaluated using numerical methods [20], [22].

Equations (5.75) and (5.78) show that convolution occurs naturally for LTI systems described by linear ordinary differential equations with constant coefficients.

### 5.7 Linearity

From equations (5.24) and (5.28) one sees why SISO LTI systems described by a linear ordinary differential equation with constant coefficients are called *linear systems*. Specifically, these equations show that the system output \( y(\cdot) \) is linear in the (joint initial condition - input) vector

\[
\begin{bmatrix}
y_0 \\
\vdots \\
(n_{1-1}) \\
y_0 \\
u(\cdot)
\end{bmatrix}.
\] (5.79)

That is, if

\[
\begin{bmatrix}
y_{0i} \\
\vdots \\
(n_{1-1}) \\
y_{0i} \\
u_i (\cdot)
\end{bmatrix} \text{ produces } y_i (\cdot) \quad i = 1, 2
\] (5.80)
then the linear combination

\[
\begin{bmatrix}
  y_0 \\
  \vdots \\
  y_{(n-1)} \\
  u(\cdot)
\end{bmatrix}
= a_1 \begin{bmatrix}
  y_{01} \\
  \vdots \\
  y_{(n-1)} \\
  u_1(\cdot)
\end{bmatrix}
+ a_2 \begin{bmatrix}
  y_{02} \\
  \vdots \\
  y_{(n-1)} \\
  u_2(\cdot)
\end{bmatrix}
\]

produces \( y(\cdot) = a_1 y_1(\cdot) + a_2 y_2(\cdot) \). (5.81)

Given this, it follows that such systems are linear in the input assuming zero initial conditions and linear in the initial conditions assuming zero input.

It should be noted that sometimes the above linearity principle is referred to as the principle of superposition \cite{23}.

*Nonlinear Systems.* Systems that are not linear are said to be *nonlinear systems* \cite{27, 47}.

**Example 5.7.1 (Saturation Nonlinearity.)**

Consider the system defined by the static input-output model

\[
y = \begin{cases}
  -1 & u < -1 \\
  u & -1 \leq u < 1 \\
  1 & 1 \leq u
\end{cases}
\]

This system defines a saturation nonlinearity. This type of nonlinearity is used, for example, to model the fact that the elevator on an aircraft can not rotate an arbitrary amount.

**5.8 Time Invariance**

For many systems the response to a given input is “time invariant,” i.e. it does not depend on when the input is applied. This notion motivates the following definition.

*Time Invariance.* Suppose that all initial conditions are set to zero and that an input \( u(\cdot) \) results in an output \( y(\cdot) \). A system is said to be *time invariant* if the delayed input \( u(\cdot - \Delta) \) results in the corresponding delayed output \( y(\cdot - \Delta) \) for any input \( u(\cdot) \) and any delay \( \Delta \geq 0 \). This definition may
be visualized via system block diagram as indicated in Figure 5.3 where $T$ represents the system in question and $S_\Delta$ is a delay operator defined by the relationship

$$(S_\Delta w)(t) \overset{\text{def}}{=} w(t - \Delta).$$  \hfill (5.83)

![Block Diagram Visualization of Time Invariance Test](image)

Figure 5.3: Block Diagram Visualization of Time Invariance Test: $(S_\Delta T u)(t) = (T S_\Delta u)(t)$

From Figure 5.3, it follows that a system $T$ is *time invariant* if and only if the operator defined by $T$ commutes with the delay operator $S_\Delta$. That is, the system $T$ is time invariant if and only if

$$y(t - \Delta) = (T u)(t - \Delta) = (S_\Delta T u)(t) = (T S_\Delta u)(t) = (T u(\cdot - \Delta))(t)$$  \hfill (5.84)

for any input $u$ and any delay $\Delta \geq 0$.

*Convolution Operators.* If $T$ defines a convolution operator, then it can be shown that $T$ defines a time invariant system (cf. equations (5.17)-(5.21)). All systems described by linear ordinary differential equations with constant coefficients define time invariant systems.

*Linear Time Invariant (LTI) Systems.* Systems which are linear and time invariant are called *linear time invariant (LTI)* systems. Practically speaking, most LTI systems define a convolution operation, an impulse response, and a transfer function. Only “pathological” (i.e. weird) LTI systems [26, pg. 3] do not define a convolution operation. All systems described by linear ordinary differential equations with constant coefficients define an LTI system.
Time Varying Systems. Systems which are not time invariant are said to be time varying systems. Linear differential equations with coefficients that depend on the independent variable $t$ are an example of linear time varying systems.

5.9 Causality

A causal system is one whose current output depends on the present and/or previous inputs. Based on this definition, it follows that the RLC circuit in Figure 5.1 is a causal system.

More general (not necessarily causal) LTI systems may be described by the convolution operation:

$$ y(t) = (h * u)(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau. \quad (5.85) $$

For more general systems, the current output may depend on future inputs as well as past and present inputs. Such descriptions might occur in applications where $t$ does not represent real time. For example, $h(\cdot)$ might represent the impulse response of a linear system whose purpose is to process a spatially distributed image $u(\cdot)$ [17].

Example 5.9.1 (A Noncausal Linear Space Invariant System.)

Consider the system described by the following second order linear ordinary differential equation with constant coefficients

$$ \ddot{y}(x) + y(x) = -2u(x) \quad x \in (-\infty, \infty) \quad (5.86) $$

where $x$ represents a spatial independent variable - not time. We seek a solution $y(\cdot)$ which depends on $u(\cdot)$ and holds for all $x \in (-\infty, \infty)$. This goal is very different from that which we have pursued in the initial value problems which we have focussed on thus far.

Suppose one defines a bilateral Laplace transform

$$ F(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} f(x)e^{-sx}dx. \quad (5.87) $$
With this definition, it can be shown that

\[ e^{-ax}1(x) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s + a} \quad Re s > -Re a \quad (5.88) \]

\[ e^{ax}1(-x) \overset{\mathcal{L}}{\leftrightarrow} \frac{-1}{s - a} \quad Re s < Re a \quad (5.89) \]

\[ f^{(n)}(x) \overset{\mathcal{L}}{\leftrightarrow} s^n F(s) \quad n = 0, 1, 2, \ldots \quad (5.90) \]

While the first transform pair is not new, the latter transform pairs do differ from the unilateral transform pairs we have seen earlier (cf. Tables 4.1, 4.2).

Applying these to our differential equations yields

\[ Y(s) = \frac{-2}{s^2 - 1} U(s). \quad (5.91) \]

From this, it follows that the system transfer function

\[ H(s) = \frac{-2}{s^2 - 1} = \frac{1}{s + 1} + \frac{-1}{s - 1} \quad -1 < Re s < 1. \quad (5.92) \]

The corresponding impulse response is given by the two-sided function

\[ h(x) = e^{-x}1(x) + e^{x}1(-x) = e^{-|x|}. \quad (5.93) \]

The output is now given by the following convolution operation

\[ y(x) = \int_{-\infty}^{\infty} h(x - z)u(z)dz \quad x \in (-\infty, \infty). \quad (5.94) \]

Here, \( H(\cdot) \) can be thought of as an linear space invariant system (filter or image processor) which processes the two-sided image intensity specified by the input \( u(\cdot) \). Because the system impulse response decays to zero exponentially as the spatial variable \( |x| \to \infty \), it is meaningful to say that the system is exponentially stable.

\[ \blacksquare \]

5.10 Bounded-Input Bounded-Output (BIBO) Stability

In an earlier section, we considered one notion of stability. Here we consider another notion.
Bounded-Input Bounded-Output (BIBO) Stability. A system is said to be bounded-input bounded-output (BIBO) stable if bounded inputs result in bounded outputs.

Suppose that \( h(\cdot) \) is the impulse response of a SISO LTI system. Given this, the LTI system defined by \( h(\cdot) \) defines a BIBO system [13, pg. 23] if and only if

\[
\|h\|_{\mathcal{L}_1} \overset{\text{def}}{=} \int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty.
\]  

For such a system the following relationship holds:

\[
\max_t |y(t)| \leq \|h\|_{\mathcal{L}_1} \max_t |u(t)|.
\]  

Similar results hold for general (not necessarily causal) multiple-input multiple-output (MIMO) LTI systems with well defined impulse response matrices and convolution representations [13, pg. 26].

### 5.11 Step Response

Once the impulse response \( h(\cdot) \) of an LTI system is known, equation (5.28) may be used to determine the step response of the system. Denoted by the symbol \( s(\cdot) \), the step response is given by

\[
s(t) = \int_0^t h(t-\tau)d\tau = \int_0^t h(\tau)d\tau
\]  

where the latter expression follows from a substitution of variables. For very complex \( h(\cdot) \) - for which closed form integral expressions are unknown - this relationship may be used to determine \( s(\cdot) \) using some numerical integration routine [20], [22]. From equation (5.97), it follows that

\[
h(t) = \frac{d}{dt}s(t).
\]  

These relationships between \( h(\cdot) \) and \( s(\cdot) \) follow from equations (3.8)-(3.9) and the fact that the systems under consideration are linear. One can also use standard partial fraction expansion techniques to determine the step response. This is illustrated in the following example.
Example 5.11.1 (Step Response For Standard Second Order System.)

Consider a SISO system whose transfer function is in standard second order form as follows:

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}. \]  (5.99)

The step response of this system can be found by finding the inverse Laplace transform of

\[ Y(s) = H(s)\frac{1}{s}. \]  (5.100)

Using partial fraction expansion techniques gives rise to several cases which depend on the damping factor \( \zeta \) as follows.

Undamped/Underdamped ( \( 0 \leq \zeta < 1 \)):

\[ s(t) = \left[ A + 2|B|e^{-\zeta \omega_n t} \cos(\omega_n \sqrt{1-\zeta^2} t + \angle B) \right] 1(t) \]  (5.101)

where \( A = H(0) = 1 \) and \( B = \frac{\omega_n^2}{s+\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2}} \)

Critically Damped (\( \zeta = 1 \)):

\[ s(t) = \left[ A + Be^{-\omega_n t} + Ce^{-\omega_n t} \right] 1(t) \]  (5.102)

where \( A = H(0) = 1, B = \omega_n^2, \) and \( C = 0; \)

Overdamped (\( \zeta > 1 \)):

\[ s(t) = \left[ A + Be^{s_1 t} + Ce^{s_2 t} \right] 1(t) \]  (5.103)

where \( A = H(0) = 1, B = \frac{\omega_n^2}{s(s_1-s_2)}, \) and \( C = \frac{\omega_n^2}{s(s_2-s_1)} \).

Each of these may be visualized as shown in Figure (5.4). It should be noted that in each case, the steady state output is given by \( s(\infty) = H(0) = 1. \)
Figure 5.4: Step Response For Standard Second Order System: Dependence on Damping Factor

Overhoot, Undershoot, Time-To-Peak. From the above discussion, one sees that for small $\zeta$, the step response of our standard second order system (5.55) exhibits oscillatory behavior. When the response goes above unity - the steady state value - we say that the response exhibits an overshoot. When it dips below unity, we say that we have an undershoot. More specifically, one observes that for small enough $\zeta$, the step response exhibits a rapid rise to a well defined maximum. In such a case, we say that the response exhibits overshoot. See Figure 5.5.

The maximum percent overshoot $M_p$ is defined as follows [32, pp. 150-156]:

$$M_p \overset{\text{def}}{=} \frac{\max_{t \geq 0} s(t) - s(\infty)}{s(\infty)}.$$  \hspace{1cm} (5.104)

It can be shown via elementary calculus that overshoot occurs for $\zeta \in [0, 1)$. Moreover,

$$M_p = e^{-\zeta \omega_n t_p} = e^{\frac{-\pi}{\sqrt{1-\zeta^2}}}$$  \hspace{1cm} (5.105)

where

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$  \hspace{1cm} (5.106)

is the time at which $y$ achieves its maximum overshoot (peak); i.e. $\max_{t \geq 0} y(t) = y(t_p)$. 
Figure 5.5: Step Response For Standard Second Order System: Visualization of Overshoot

Equation (5.105) demonstrates that $M_p$ depends on $\zeta$, is independent of $\omega_n$, and lies in the interval $M_p \in (0, 1]$ for $\zeta \in [0, 1)$. Specifically, $M_p$ approaches unity as $\zeta$ approaches zero (undamped system). $M_p$ approaches zero as $\zeta$ approaches unity (critically damped system). Equation (5.105) for $M_p$ may be visualized as shown in Figure 5.5. Equation (5.106) demonstrates that the peak time $t_p$ varies inversely with the damped natural frequency $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. As $\omega_d$ is increased, the time to reach the peak $t_p$ decreases.

Example 5.11.2 (Effect Of Numerator Zero On Step Response Of Standard Second Order System.)

Consider the underdamped SISO LTI system

$$H(s) = \frac{2}{a} \left[ \frac{s + a}{s^2 + 2s + 2} \right].$$

(5.107)

with poles at $s = -1 \pm j1$ and a zero at $s = -a$. We would like to understand how the numerator zero affects the step response of the system. The system’s step response is plotted in Figure 5.6.
for different values of the parameter $a$. For large $a$, the system responds like the underdamped ($\zeta = 0.7071$) second order system

$$H(s) = \frac{2}{s^2 + 2s + 2}.$$  \hspace{1cm} (5.108)

As $a$ is decreased, the derivative action of the zero becomes more pronounced and the system’s step response exhibits more overshoot. The derivative action becomes most pronounced when the numerator zero lies to the right of the real part of the system pole; i.e. $a < 1$.

---

Figure 5.6: Step Response For Standard Second Order System: Effect Of Numerator Zero

### 5.12 Sinusoidal Steady State Analysis And System Zeros

Sinusoidal signals are arguably amongst the most widely used test signals in the analysis of dynamical systems. Understanding their steady state impact on LTI system is of fundamental importance.

*Steady State Response To An Exponential.* Let $H$ denote the transfer function of an asymptotically stable system (e.g. RLC series circuit in Figure 5.1). Suppose that $s_o$ denotes an arbitrary complex
number in the region of convergence of \( H(\cdot) \) and that the input

\[
u(t) = e^{s_o t}
\]

is applied to \( H \). The output \( y(\cdot) \) is given by

\[
y(t) = y_{\text{air}}(t) + (u * h)(t) = y_{\text{air}}(t) + \int_0^t u(t - \tau)h(\tau)d\tau = y_{\text{air}}(t) + \int_0^t e^{s_o(t-\tau)}h(\tau)d\tau
\]

or

\[
y(t) = y_{\text{air}}(t) + \left[ \int_0^t h(\tau)e^{-s_o \tau}d\tau \right] e^{s_o t}
\]

For \( t \) large, \( y \) approaches the \textit{steady state} function

\[
y_{\text{ss}} = H(s_o) e^{s_o t}.
\]

\textit{Steady State Response To An Exponential Sinusoid.} In what follows, it will be seen how this fundamental relationship establishes a foundation for the study of LTI systems. Toward this end, suppose that

\[
u(t) = |X|e^{\sigma o t} \cos(\omega_o t + \angle X) = \left| \frac{X}{2} \right| [e^{(\sigma_o + j\omega_o)t}e^{j\angle X} + e^{(\sigma_o - j\omega_o)t}e^{-j\angle X}].
\]

This produces

\[
y_{\text{ss}} = \left| \frac{X}{2} \right| \overline{H(\sigma_o + j\omega_o)}e^{(\sigma_o + j\omega_o)t}e^{j\angle X} + \left| \frac{X}{2} \right| \overline{H(\sigma_o - j\omega_o)}e^{(\sigma_o - j\omega_o)t}e^{-j\angle X}
\]

Using the fact that \( \overline{H(s_o)} = H(\overline{s_o}) \) when \( h(\cdot) \) is real, yields the following:

\[
y_{\text{ss}} = \text{Re} \left[ \overline{H(\sigma_o + j\omega_o)}|X|e^{j\angle X}e^{(\sigma_o + j\omega_o)t} \right]
\]

or

\[
y_{\text{ss}} = \overline{H(\sigma_o + j\omega_o)}|X| e^{\sigma_o t} \cos(\omega_o t + \angle X + \angle H(j\omega_o)).
\]

This result implies that exponential sinusoidal inputs result in a steady state exponential sinusoid.
Steady State Response To A Sinusoid. From the above relationship, one obtains the main result for sinusoidal steady state analysis of LTI systems:

$$u(t) = |X| \cos(\omega_0 t + \angle X) \xrightarrow{\text{produces}} y_{ss} = |H(j\omega_0)||X| \cos(\omega_0 t + \angle X + \angle H(j\omega_0)).$$  \hspace{1cm} (5.117)

From this result, it follows that the steady state output of an LTI system to a sinusoidal input is a sinusoid. The input sinusoid is modified in amplitude by the multiplicative factor $|H(j\omega_0)|$ and in phase by the additive factor $\angle H(j\omega_0)$. Equation (5.117) arguably establishes the main result for the study of LTI systems.

Interpretation of Zeros. The zeros of a transfer function are those values of $s$ such that $H(s) = 0$. From this, it follows that if $H(\cdot)$ represents an asymptotically stable LTI system with zeros at $s = \pm j\omega_0$, then the application of an input $u(t) = |X| \cos(\omega_0 + \angle X)$ will result in a steady state output $y_{ss} = 0$ since $H(j\omega_0) = 0$.

Steady State DC Analysis and DC Gain. A special case of equation (5.117) arises when the input $u$ is a constant. Such an input corresponds to an input frequency $\omega_0 = 0$. In such a case, we say that the input $u$ is a dc signal. The term “dc” is widely used and has its roots in circuit analysis (i.e. dc for direct current [23]). For such a signal, equation (5.117) becomes

$$u(t) = k \xrightarrow{\text{produces}} y_{ss} = H(0) k$$  \hspace{1cm} (5.118)

for any real constant $k$. The quantity $H(0)$ is called the dc gain of the system. Since many systems are designed to operate at low frequencies, it follows that the above dc analysis is of great importance in the analysis and design of systems.
5.13 Frequency Response, Magnitude Response, Phase Response, And Fourier Transform

In studying the steady state response of LTI systems to sinusoidal signals, it becomes immediately evident that the system transfer function $H$, and specifically, the complex function of $\omega$

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)} \quad (5.119)$$

is of fundamental importance.

**Frequency Response, Magnitude Response, Phase Response.** The two functions $|H(j\omega)|$ and $\angle H(j\omega)$ are referred to as the frequency response of the system. Sometimes $H(j\omega)$ is called the system frequency response or frequency response. Sometimes $H(j\omega)$ is called the spectrum of the signal $h(\cdot)$. The function $|H(j\omega)|$ is called the magnitude response of the system. The function $\angle H(j\omega)$ is called the phase response of the system.

**Magnitude and Phase Response Properties.** It can be shown that if the impulse response $h(\cdot)$ is real (i.e. $\overline{h}(\cdot) = h(\cdot)$) - which is typically the case in most applications, then

$$|H(j\omega)| = |H(-j\omega)| \quad (5.120)$$

$$\angle H(j\omega) = -\angle H(-j\omega). \quad (5.121)$$

That is, $|H(j\omega)|$ is an even function of $\omega$ and $\angle H(j\omega)$ is an odd function of $\omega$.

**Continuous Time Fourier Transform.** The unilateral continuous time Fourier transform of $h(\cdot)$ [36] is defined as follows:

$$(\mathcal{F}h)(j\omega) \overset{\text{def}}{=} \int_{0-}^{\infty} h(\tau)e^{-j\omega \tau} d\tau. \quad (5.122)$$

It is important to note that this function is not always equal to the function obtained by substituting $s = j\omega$ into the Laplace transform of $h(\cdot)$ [36]. This is because $s = j\omega$ need not lie within the
region of convergence of $H(\cdot)$. In general,

$$
(\mathcal{L}h)(s)|_{s=j\omega} = (\mathcal{F}h)(j\omega)
$$

(5.123)

if and only if $s = j\omega$ lies within the region of convergence of $(\mathcal{L}h)(\cdot)$. The above statements also apply when working with bilateral Laplace and Fourier transforms:

$$
(\mathcal{L}h)(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau
$$

(5.124)

$$
(\mathcal{F}h)(j\omega) \overset{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau.
$$

(5.125)

**System Bandwidth.** It is often useful to discuss a system’s ability or inability to amplify or attenuate rapidly changing input signals. The concept of bandwidth attempt to quantify this notion. The most widely accepted definition for bandwidth is one which applies to LTI systems with a low pass magnitude response; i.e. one which is approximately zero above some frequency. The bandwidth, denoted BW, of an LTI system with transfer function $H(\cdot)$ is defined to be that frequency at which the magnitude response is down 3 db (a factor of $1/\sqrt{2}$) from the dc gain; i.e.

$$
|H(j\text{BW})| = \frac{1}{\sqrt{2}} |H(j0)|.
$$

(5.126)

If a system is to amplify rapidly varying signals, then it should have a high bandwidth. Scaling a system’s input or output by a gain greater than unity increases the bandwidth of the system. Scaling a system’s input or output by a gain less than unity decreases the bandwidth of the system.

**Example 5.13.1 (Magnitude And Phase For Standard Second Order System.)**

The dependence of the magnitude and phase responses for the standard second order system

$$
H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
$$

(5.127)

on the damping factor $\zeta$ are shown in Figure 5.7.
Magnitude and Phase Response ($\omega_n = 1, \zeta = 0.1, 0.2, 0.3, 0.5, 0.7071$)

Figure 5.7: Frequency Response For Standard Second Order System: Dependence on Damping Factor

In this figure, the independent variable $\omega$ is plotted on a logarithmic (base 10) horizontal scale. The magnitude response $|H(j\omega)|$ is plotted on a linear vertical scale in decibels (dB); i.e. $20 \log_{10} |H(j\omega)|$ is plotted on a linear vertical scale [32]. The angle response $\angle H(j\omega)$ is plotted in degrees on a linear vertical scale. Methods for systematically sketching asymptotic approximations - called Bode plots - to these were developed by H.W. Bode [5]. A discussion of Figure 5.7 now follows.

We know that for small $\zeta$, our standard second order system has poles near the imaginary axis. For $\zeta = 0$ both poles are on the imaginary axis at $s = \pm j\omega$ and $H(\pm j\omega) = \infty$. Based on this, one expects by continuity that the magnitude response will exhibit a large peak for small $\zeta$. If one lets $\omega_r$ denote the frequency at which the magnitude response peaks and $M_r$ denote the peak magnitude; i.e.

$$M_r \overset{\text{def}}{=} \max_{\omega} |H(j\omega)| = |H(j\omega_r)|,$$

then one obtains the following relationships for $\omega_r$ and $M_r$ [32, pp. 482-484]:

$$\omega_r = \begin{cases} \omega_n \sqrt{1 - 2\zeta^2} & \zeta \in [0, \frac{1}{\sqrt{2}}] \\ 0 & \zeta > \frac{1}{\sqrt{2}} \end{cases}$$ (5.129)
$$M_r = \begin{cases} 
\frac{1}{2\zeta \sqrt{1-\zeta^2}} & \zeta \in [0, \frac{1}{2}] \\
1 & \zeta > \frac{1}{2}.
\end{cases} \tag{5.130}$$
Chapter 6

Introduction To Bode Plots

(1) Consider the SISO LTI system

\[ L = \frac{1}{s}. \]  \hspace{1cm} (6.1)

This system represents an integrator. The magnitude and phase response for \( L \) is plotted in Figure 6.1. The slope of the magnitude plot is \(-20 \, \text{db/dec} \) or \(-6 \, \text{db/oct} \) (\text{db} - \text{decibels}; \text{dec} - \text{decade}; \text{oct} - \text{octave}). The phase contributes \(-90 \, \text{degrees} \) at all frequencies.

![Frequency Response for an Integrator: \( L = 1/s \)]

Figure 6.1: Frequency Response For Integrator: \( L = \frac{1}{s} \).
(2) Consider the SISO LTI system

\[ L = \frac{1}{s + 0.4}. \]  

(6.2)

This system represents an approximate integrator. The magnitude and phase response for \( L \) is plotted in Figure 6.2. It approximates an integrator well at high frequencies. It approximates the integrator well above \( \omega = 4 \text{ rad/sec} \). Above \( \omega = 4 \text{ rad/sec} \), the magnitude response rolls-off at a slope of \(-20 \text{ db/dec}\).

![Integrator L = 1/s and Approximate Integrator L = 1/(s+0.4)](image)

Figure 6.2: Frequency Response For Integrator \( L = \frac{1}{s} \) and Approximate Integrator \( L = \frac{1}{s+0.4} \).
(3) Consider the SISO LTI system

\[ L_3 = s. \]  \hspace{1cm} (6.3)

This system represents a differentiator. The magnitude and phase response for \( L \) is plotted in Figure 6.3. The magnitude response has a slope of 20 \( db/dec. \) The phase response contributes 90 degrees at all frequencies.

![Frequency Response for a Differentiator: \( L = s \)]

Figure 6.3: Differentiator: \( L = s. \)
(4) Consider the SISO LTI system

\[ L_4 = \frac{10s}{s + 10} \]  

This system represents an approximate differentiator. The magnitude and phase response for \( L \) is plotted in Figure 6.4. It approximates the differentiator well at low frequencies. Below \( \omega = 1 \text{ rad/sec} \), it approximates the differentiator well.

![Differentiator L = s and Approximate Differentiator L = 10 s/(s + 10)](image)

Figure 6.4: Differentiator \( L = s \) and Approximate Differentiator \( L = \frac{10s}{s + 10} \)
(5) Consider the SISO LTI system

\[ L = \begin{bmatrix} \frac{1}{s(s+1)} \\ \frac{10^3}{s + 10^3} \end{bmatrix}. \]  

(6.5)

This system represents a second order lag-integral system with a high frequency pole. The magnitude and phase response for \( L \) are plotted in Figure 6.5. The high frequency pole contributes an additional 90 degrees of phase lag by \( \omega \approx 10^4 \text{ rad/sec} \).

\[ L = \frac{1}{s}, \quad L = \frac{1}{s} \left[ \frac{1}{s+1} \right], \quad L = \frac{1}{s} \left[ \frac{10^3}{s+1} \right] \left[ \frac{10^3}{s+10^3} \right] \]

Figure 6.5: 3rd Order System \( L = \left[ \frac{1}{s(s+1)} \right] \left[ \frac{10^3}{s+10^3} \right] \).
(6) Consider the SISO LTI system

$$L = \left[ \frac{1}{s(s+1)} \right] \left[ \frac{10^3}{s + 10^3} \right] \left[ \frac{s + 10^6}{10^6} \right].$$

This system represents a second order lag-integral system with a high frequency pole and zero. The magnitude and phase response for $L$ is plotted in Figure 6.6. The high frequency zero contributes an additional 90 degrees of phase lead by $\omega \approx 10^7$ rad/sec.

$$L = 1/s \left[ \frac{1}{s + 1} \right] \left[ \frac{10^3}{s + 10^3} \right] \text{ and } L = 1/s \left[ \frac{1}{s + 1} \right] \left[ \frac{10^3}{s + 10^3} \right] \left[ \frac{s + 10^6}{10^6} \right]$$

Figure 6.6: 3rd Order System With Numerator Zero $L = \left[ \frac{1}{s(s+1)} \right] \left[ \frac{10^3}{s + 10^3} \right] \left[ \frac{s + 10^6}{10^6} \right]$. 
(7) Consider the SISO LTI system

\[ L = \frac{1}{s(s+1)} \left[ \frac{10^3}{s + 10^3} \right] \left[ \frac{s + 10^6}{10^6} \right]^2. \] (6.7)

This system represents a second order lag-integral system with a high frequency pole and two zeros. The magnitude and phase response for \( L \) is plotted in Figure 6.7. The two high frequency zeros contribute an additional 180 degrees of phase lead by \( \omega \approx 10^7 \text{ rad/sec} \).

\[ L = \frac{1}{s} \left[ \frac{1}{s+1} \right] \left[ \frac{10^3}{s + 10^3} \right] \text{ and } L = \frac{1}{s} \left[ \frac{1}{s+1} \right] \left[ \frac{10^3}{s + 10^3} \right] \left[ \frac{s + 10^6}{10^6} \right]^2 \]

Figure 6.7: 3rd Order System With Two Numerator Zeros: \( L = \frac{1}{s(s+1)} \left[ \frac{10^3}{s + 10^3} \right] \left[ \frac{s + 10^6}{10^6} \right]^2. \)
(8) Consider the SISO LTI system

\[ L = \frac{5}{s - 1} \]  

(6.8)

This system represents a first order unstable system. The magnitude and phase response for \( L \) is plotted in Figure 6.8. While the magnitude response is identical to that of the stable system \( \frac{5}{s+1} \), the phase response looks very different. The phase contributes 180 degrees of phase lag at dc. As we go up in frequency, \( L \) contributes phase lead.

Figure 6.8: Frequency Response For Unstable First Order System: \( L = \frac{5}{s - 1} \).
(9) Consider the SISO LTI system

\[
L = \begin{bmatrix}
\frac{10}{s - 1} \\
\frac{10^3}{s + 10^3}
\end{bmatrix}.
\]

This system represents a first order unstable system with a high frequency pole. The magnitude and phase response for \( L \) is plotted in Figure 6.9. The high frequency pole contributes an additional 90 degrees of lag by \( \omega \approx 10^4 \text{ rad/sec} \).

\[
L = \frac{10}{s - 1} \quad \text{and} \quad L = \left[ \frac{10}{s - 1} \right] \frac{10^3}{s + 10^3}.
\]
(10) Consider the SISO LTI system

\[ L = \begin{bmatrix} \frac{10}{s-1} \\ \frac{10^3}{s+10^3} \\ \frac{10^6}{s+10^6} \end{bmatrix}. \]  

(6.10)

The magnitude and phase response for \( L \) is plotted in Figure 6.10. The high frequency pole contributes an additional 90 degrees of lag by \( \omega \approx 10^7 \) rad/sec.

\[ L = \left[ \frac{10}{(s - 1)} \right] \frac{10^3}{(s+10^3)} \quad \text{and} \quad L = \left[ \frac{10}{(s - 1)} \right] \left[ \frac{10^3}{(s+10^3)} \right] \frac{10^6}{(s + 10^6)} \]

![Graph showing magnitude and phase response](image)

**Figure 6.10:** \( L = \begin{bmatrix} \frac{10}{s-1} \\ \frac{10^3}{s+10^3} \\ \frac{10^6}{s+10^6} \end{bmatrix}. \)
Chapter 7

An Introduction To State Space Concepts

7.1 State Of A System And State Variables

State Of A System. The state of a system is the minimum information required to predict the system’s future output $y(\cdot)$ given knowledge of current and future inputs $u(\cdot \cdot \cdot \cdot \cdot)$.

Example 7.1.1 (State and State Variables For Series RLC Circuit.)

Consider the series RLC circuit in Figure 5.1. From equation (5.3), it follows that the state of the circuit at $t = 0$ (or at any initial time $t = t_0$) is the initial condition vector $[y_0 \ y_0]$. Because of this, we say that $y$ and $\dot{y}$ are state variables. It must be noted that the state of a system is not unique. For the RLC circuit it suffices, for example, to know any two independent pieces of information (e.g. initial capacitor voltage and inductor voltage) in order to determine $[y_0 \ y_0]$ and predict the future output with knowledge of future inputs. Given this, it follows that state variables are also not unique. For the RLC circuit, another set of state variables consists of the capacitor voltage $y$ and the inductor current $i_L$.

Selection Of State Variables. In general, state variables are typically selected to be variables associated with energy storage (e.g. velocities, spring displacements, etc.).
For RLC circuits, the capacitor voltages and inductor currents

are typically chosen as state variables [23].

Any \(n^{th}\) order system has \(n\) (not unique) state variables associated with it. Similarly, any system
which is described by \(n\) state variables may be represented by an \(n^{th}\) order (vector) differential
equation. This will be further explored below.

7.2 Introduction to Descriptor Models, State Space Models, and
MIMO LTI Systems

Many dynamical systems are described by very high order differential equations (e.g. a flexible
space structure [25]). Rather than writing high order differential equations, it is often much easier
to write many simple equations. This motivates the new system descriptions presented in this
section.

Descriptor Representation. Consider systems which are described by linear algebraic and ordinary
differential equations with constant coefficients. Such systems may be written in descriptor form
as follows:

\[
M \dot{x} = Nx + Pu \quad x(0^-) = x_0 \quad t > 0 
\]

(7.1)

\[
y = Cx + Du 
\]

(7.2)

where \(M \in \mathbb{R}^{n \times n}\), \(N \in \mathbb{R}^{n \times n}\), \(P \in \mathbb{R}^{n \times m}\), \(u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m\) denotes a vector of \(m\) input
variables, \(x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n\) denotes a vector of \(n\) state variables, \(y = [y_1, \ldots, y_p]^T \in \mathbb{R}^p\)
denotes a vector of \(p\) output variables, and \(x_0 = [x_{1_0}, \ldots, x_{n_0}]^T \in \mathbb{R}^n\) denotes a vector of \(n\) initial
conditions for the state.

For systems described by linear differential and algebraic equations, it is always easy to write
system equations in descriptor form.
State Space Representation: Multiple-Input Multiple-Output (MIMO) LTI Systems. When the matrix $M$ is invertible, equation (7.1) may be rewritten in so-called state space form. Doing so yields:

State Equation: $\dot{x} = Ax + Bu; \quad x(0^-) = x_0 \quad t > 0 \quad (7.3)$

Output Equation: $y = Cx + Du. \quad (7.4)$

where

$$A = M^{-1}N \quad (7.5)$$

$$B = M^{-1}P. \quad (7.6)$$

For general MIMO LTI systems, $A \in \mathcal{R}^{n \times n}, B \in \mathcal{R}^{n \times m}, C \in \mathcal{R}^{p \times n}, D \in \mathcal{R}^{p \times m}, u = [u_1, \ldots, u_m]^T \in \mathcal{R}^m$ denotes a vector of $m$ input variables, $x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n$ denotes a vector of $n$ state variables, $y = [y_1, \ldots, y_p]^T \in \mathcal{R}^p$ denotes a vector of $p$ output variables, and $x_0 = [x_{1_o}, \ldots, x_{n_o}]^T \in \mathcal{R}^n$ denotes a vector of $n$ initial conditions for the state. The quadruple $(A, B, C, D)$ is said to define a state space representation for the system. The pair $(A, B)$ define the state equation (7.3) while the pair $(C, D)$ define the output equation (7.4). Because the state equation contains $n$ differential equations, we say that the system is an $n^{th}$ order system. A general MIMO LTI system with state space representation $(A, B, C, D)$ may be visualized as shown in Figure 7.1.

![Figure 7.1: Block Diagram Visualization of General LTI System State Space Representation](image_url)
Utility of State Space Representations. State space representations \((A, B, C, D)\) are very useful for a variety of reasons:

- Many systems may be approximated well by a state space representation \((A, B, C, D)\);
- A rich set of analysis and design techniques exist;
- State-of-the-art support software is readily available [11], [21], [30].

Example 7.2.1 (Series RLC Circuit Descriptor and State Space Representations.)

Consider the series RLC circuit in Figure 5.1. It may be represented by two first order linear differential equations as follows:

\[
\begin{align*}
    u &= Ri_L + L \frac{di_L}{dt} + y \\
    i_L &= C \frac{dy}{dt}
\end{align*}
\]

(7.7) (7.8)

These may be written in so called descriptor form as follows:

\[
\begin{align*}
    L \frac{di_L}{dt} &= -y - Ri_L + u \\
    C \frac{dy}{dt} &= i_L
\end{align*}
\]

(7.9) (7.10)

or in matrix-vector descriptor form as follows:

\[
M \dot{x} = Nx + Pu
\]

(7.11)

where

\[
M = \begin{bmatrix} 0 & L \\ C & 0 \end{bmatrix}, \quad N = \begin{bmatrix} -1 & -R \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} y \\ i_L \end{bmatrix}
\]

(7.12)

Since the matrix \(M\) is invertible, the above equation may be written in state space form as follows:

\[
\dot{x} = Ax + Bu
\]

(7.13)
where
\[
A = M^{-1}N = \begin{bmatrix}
0 & \frac{1}{\tau} \\
-\frac{1}{L} & -\frac{R}{L}
\end{bmatrix}
B = M^{-1}P = \begin{bmatrix}
0 \\
\frac{1}{L}
\end{bmatrix}.
\] (7.14)

An output equation may be written as follows:
\[
y = Cx + Du
\] (7.15)

where
\[
C = \begin{bmatrix}
1 \\
0
\end{bmatrix}
D = \begin{bmatrix}
0
\end{bmatrix}.
\] (7.16)

7.3 Linear Transformation of System Variables

In many applications, it is desirable to

- change the units of the variables being used or
- change the coordinate system being used.

In either case, a transformation of variables is required. Given this, suppose that we are given an LTI system with state space representation \((A_1, B_1, C_1, D_1)\):
\[
\dot{x}_1 = A_1 x_1 + B_1 u_1
\] (7.17)
\[
y_1 = C_1 x_1 + D_1 u_1.
\] (7.18)

Suppose that we desire to transform the variables \(u_1, x_1, y_1\). This can be accomplished via the following linear transformations:
\[
u_2 = S_u u_1
\] (7.19)
\[
x_2 = S_x x_1
\] (7.20)
\[
y_2 = S_y y_1
\] (7.21)
where $S_u \in \mathbb{R}^{m \times m}$, $S_x \in \mathbb{R}^{n \times n}$, $S_y \in \mathbb{R}^{p \times p}$ are invertible (nonsingular) matrices. Given this, the new state space representation $(A_2, B_2, C_2, D_2)$ is given by:

$$
\begin{align*}
\dot{x}_2 &= A_2 x_2 + B_2 u_2 \\
y_2 &= C_2 x_2 + D_2 u_2
\end{align*}
$$

(7.22) (7.23)

where

$$
A_2 = S_x A_1 S_x^{-1} \quad B_2 = S_x B_1 S_u^{-1} \quad C_2 = S_y C_1 S_x^{-1} \quad D_2 = S_y D_1 S_u^{-1}.
$$

(7.24) (7.25)

The transformation matrix $S_x$ is often called a similarity transformation [44] and the matrices $A_1$ and $A_2$ are said to be similar to one another. As might be expected, it can be shown that changing units does not change the fundamental properties of a system.

### 7.3.1 State Space Representation Analysis Via Laplace: System Resolvent

The matrix-vector equations (7.3) and (7.4) may be analyzed using Laplace Transforms. Taking the transform of each equation yields:

$$
\begin{align*}
X(s) &= (sI - A)^{-1} x_0 + (sI - A)^{-1} B U(s) \\
Y(s) &= C (sI - A)^{-1} x_0 + [ C (sI - A)^{-1} B + D ] U(s)
\end{align*}
$$

(7.26) (7.27)

The matrix $(sI - A)^{-1}$ is called the resolvent of the system.

### 7.4 Transfer Function Matrix, Matrix Exponential, And Impulse Response Matrix

For SISO systems, the concept of a transfer function is important. For MIMO systems, the analogous concept is the transfer function matrix. From equation (7.27), one sees that the transfer
function matrix $H(\cdot)$ from the input vector $u = [u_1, \ldots, u_m]^T$ to the output vector $y = [y_1, \ldots, y_p]^T$ is given by

$$H(s) = C(sI - A)^{-1}B + D. \tag{7.28}$$

$H(\cdot)$ is a $p \times m$ matrix of scalar transfer functions. Its $ij^{th}$ entry represents the transfer function from the $j^{th}$ input $u_j(\cdot)$ to the $i^{th}$ output $y_i(\cdot)$. At this point, it is useful to identify the matrix exponential $e^{At}$ as the inverse Laplace transform of $(sI - A)^{-1}$; i.e.

$$e^{At} \text{ def } \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \mathcal{L}^{-1}\{ (sI - A)^{-1} \}. \tag{7.29}$$

This implies that the matrix exponential $e^{At}$ may be computed as the term-by-term inverse Laplace transform of $(sI - A)^{-1}$. Given the above, the impulse response matrix $h$ is given by

$$h(t) = Ce^{At}B1(t) + D\delta(t) \tag{7.30}$$

and the general solutions for $x(\cdot)$ and $y(\cdot)$ are given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \quad \text{(7.31)}$$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \quad \text{(7.32)}$$

**Linearity.** Equations (7.31)-(7.32) show that both the state $x(\cdot)$ and the output $y(\cdot)$ are linear in the vector $[x_o^T \quad u^T(\cdot)]^T$ - a confirmation that systems described by a state space representation $(A, B, C, D)$ do indeed define a linear system.

**Time Invariance.** Because the integral in equation (7.32) defines a convolution operator, it follows that systems described by a state space representation $(A, B, C, D)$ do indeed define a time invariant system. See equations (5.17)-(5.21).

**Computational Approaches.** Several approaches exist for evaluating $x(\cdot)$ and $y(\cdot)$. Two commonly used approaches are (i) s-domain matrix algebra [44] with term-by-term Laplace transform inversion...
and (ii) eigenvalue-eigenvector techniques - to be discussed subsequently. While the first approach is a straightforward extension of classical SISO techniques, the second method offers new useful insight regarding the natural modes of a system. Numerical integration [22], of course, represents one more approach.

Structure And Terminology: State Transition Matrix. If the input \( u(\cdot) \) is applied at \( t = t_o \), then equation (7.31) may be written as

\[
x(t) = \Phi(t, t_o)x_o + \int_{t_o}^{t} \Phi(t, \tau)Bu(\tau)d\tau
\]

(7.33)

where \( x(t_o) = x_o \) and

\[
\Phi(t, \tau) \overset{\text{def}}{=} e^{A(t-\tau)}
\]

(7.34)

is called the state transition matrix for the system.

Properties of State Transition Matrix. It can be shown that the state transition matrix satisfies the following properties:

\[
\Phi(t, t) = I \quad \text{(Identity Property)}
\]

(7.35)

\[
\Phi(t_1, t_3) = \Phi(t_1, t_2)\Phi(t_2, t_3) \quad \text{(Semi–Group Property)}
\]

(7.36)

\[
\frac{d}{dt}\Phi(t, \tau) = A\Phi(t, \tau) \quad \text{(Derivative Property)}
\]

(7.37)

for all \( t, t_1, t_2, t_3 \in \mathcal{R} \).

Looking Ahead To Time Varying Systems. In subsequent sections, time varying systems will be studied in greater detail. Given this, it is useful to note that

- The structure exhibited in equation (7.33) will be seen to hold when the matrices \((A, B)\) are functions of the independent variable \( t \);
Properties (7.35)-(7.37) will hold when the matrix $A$ is a function of the independent variable $t$. In fact, it can be shown [6, pg. 33] that if

$$A(t) \left[ \int_{t_0}^{t} A(\sigma) d\sigma \right] = \left[ \int_{t_0}^{t} A(\sigma) d\sigma \right] A(t)$$

for all $t$, then

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} A(\sigma) d\sigma}.$$  (7.39)

### 7.5 Obtaining A State Space Representation For A SISO Input-Output (Transfer Function) Description

It is often useful to describe a system with a given transfer function (input-output model), by a state space model. To demonstrate how this can be done, consider the following arbitrary 3rd order transfer function:

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} + d$$  (7.40)

where the $a_i$, $b_i$, and $d$ are arbitrary real (scalar) constants. The goal is to obtain a 3rd order state space representation $(A, B, C, D)$ whose transfer function is $H(\cdot)$. To do this, first note that $H$ may be rewritten as follows:

$$H(s) = \frac{\frac{b_2}{s} + \frac{b_1}{s^2} + \frac{b_0}{s^3}}{1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_0}{s^3}} + d$$  (7.41)

Now consider the block diagram shown in Figure 7.2. In this figure, the integrator states (their outputs) have been labeled $x_1$, $x_2$, and $x_3$, respectively. From this figure, one can “walk around” the diagram to obtain the following s-domain (zero initial condition) relationships:

$$\dot{x}_3 = u - \left[ a_2 x_3 + a_1 x_2 + a_0 x_1 \right]; \quad x_1 = \frac{\dot{x}_3}{s^3}, \quad x_2 = \frac{\dot{x}_3}{s^2}, \quad x_3 = \frac{\dot{x}_3}{s}$$

(7.42)

$$\dot{x}_3 = u - \left[ \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_0}{s^3} \right] \dot{x}_3$$

(7.43)

$$\dot{x}_3 = \frac{1}{1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_0}{s^3}} u$$

(7.44)

$$y = \left[ \frac{b_2}{s} + \frac{b_1}{s^2} + \frac{b_0}{s^3} \right] \dot{x}_3 + d u$$

(7.45)
\[
\begin{aligned}
\frac{b_2}{s^2} + \frac{b_1}{s} + \frac{b_0}{s} &= \frac{1}{s^3 + \frac{a_2}{s^2} + \frac{a_1}{s} + \frac{a_0}{s^3}} \quad \text{u + d \ u} \\
&= H(s) \ u.
\end{aligned}
\] 

(7.46)  

(7.47)

This shows that the 3rd order system shown in Figure 7.2 has the desired transfer function.

![Figure 7.2: General 3rd Order Transfer Function: Obtaining a State Space Representation](image)

From Figure 7.2, one immediately obtains the following state space representation:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a_o & -a_1 & -a_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\ u
\]

(7.48)

\[
y = \begin{bmatrix}
b_o & b_1 & b_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ 
\begin{bmatrix}
d
\end{bmatrix}
\ u
\]

(7.49)

for the general 3rd order SISO transfer function

\[
H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} + d.
\]

(7.50)

The above state space representation is said to be in controller canonical form [26]. Its ‘A’ matrix, specifically, is said to be in companion form [26]. Other canonical forms (i.e. standard block diagram structures) are discussed in [26]. The ideas presented in this example may be extended to higher order systems and to MIMO LTI systems [26] as well.
7.6 Obtaining An Input-Output (Transfer Function Matrix) Description From An LTI State Space Representation

Given a MIMO LTI system with state space representation

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du,
\end{align*}
\]

we would like to find an input-output description.

Markov Parameters. To do this, it is helpful to define the so called Markov parameters as follows:

\[
M_k \overset{\text{def}}{=} \begin{cases} 
0 & k = -1, -2, \ldots \\\nD & k = 0 \\
CA^kB & k = 1, 2, \ldots 
\end{cases}
\]

Without loss of generality, suppose that \(A\) has an \(n\)th order characteristic polynomial given by

\[
\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0
\]

where the \(a_i\) are real (scalar) constants.

**Cayley-Hamilton Theorem.** Next, it is important to recall the Cayley-Hamilton Theorem [44].

This fundamental result from linear algebra states that every matrix satisfies its characteristic polynomial; i.e.

\[
A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I.
\]

Using the Cayley-Hamilton Theorem, one can show that the output vector \(y \in \mathcal{R}^p\) is related to the input vector \(u \in \mathcal{R}^m\) as follows:

\[
\begin{align*}
(n) & \quad y(t) + a_{n-1} y(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = B_n u(t) + B_{n-1} u(t) + \cdots + B_1 \dot{u}(t) + B_0 u(t)
\end{align*}
\]

where

\[
B_k = M_{n-k} + \sum_{i=k}^{n-1} a_i M_{i-k}.
\]
Example 7.6.1 (A Two-Input Two-Output (TITO) RLC Circuit.)

Consider the RLC circuit in Figure 7.3. The circuit, or system, inputs (forcing functions) are the voltage sources \( u_1 \) and \( u_2 \). The circuit, or system, outputs are the voltage measurements \( y_1 \) and \( y_2 \). The circuit, or system, state variables are the capacitor voltage \( x_1 \) and the inductor current \( x_2 \).

![RLC Circuit Diagram](image)

Figure 7.3: Two-Input Two-Output (TITO) RLC Circuit

In this two-input two-output (TITO) circuit, the resistor, inductor, and capacitor values are denoted \( R \), \( L \), and \( C \) respectively. It is assumed that they are nonnegative and constant.

Using impedance methods, one obtains the following s-domain zero initial condition relationships between the s-domain functions \( Y_1 \), \( Y_2 \), \( U_1 \), and \( U_2 \):

\[
Y_1(s) = \left[ \frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \right] U_1(s) + \left[ \frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \right] U_2(s). 
\]  
(7.58)

\[
Y_2(s) = Y_1(s) - U_2(s) = \left[ \frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \right] U_1(s) + \left[ \frac{-s^2 + \frac{1}{RC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \right] U_2(s). 
\]  
(7.59)

From these, one can obtain an input-output model relating \( y_1 \), \( y_2 \), \( u_1 \), and \( u_2 \). More specifically, one obtains the following second order ordinary differential equations with constant coefficients:

\[
\ddot{y}_1 + \frac{1}{RC} \dot{y}_1 + \frac{1}{LC} y_1 = \frac{1}{LC} u_1 + \frac{1}{RC} \dot{u}_2. 
\]  
(7.60)

\[
\ddot{y}_2 + \frac{1}{RC} \dot{y}_2 + \frac{1}{LC} y_2 = \frac{1}{LC} u_1 - \dot{u}_2 - \frac{1}{LC} u_2. 
\]  
(7.61)

The state space representation which relates \( u_1 \), \( u_2 \), \( x_1 \), \( x_2 \), \( y_1 \), and \( y_2 \) is as follows:

\[
\dot{x} = Ax + Bu 
\]  
(7.62)

\[
y = Cx + Du 
\]  
(7.63)
where

\[
A = \begin{bmatrix}
-\frac{1}{RC} & \frac{1}{C} \\
-\frac{1}{T} & 0
\end{bmatrix}, \quad
B = \begin{bmatrix}
0 \\
\frac{1}{T}
\end{bmatrix}, \quad
C = \begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix}, \quad
D = \begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix}
\] (7.64)

The system transfer function matrix is given by:

\[
H(s) = C(sI - A)^{-1}B + D = \begin{bmatrix}
\frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{T}} & \frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{T}} \\
-\frac{(s^2 + \frac{1}{RC}s + \frac{1}{T})}{s^2 + \frac{1}{RC}s + \frac{1}{T}} & \frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{T}}
\end{bmatrix}
\] (7.65)

This is expected from equations (7.58) and (7.59).

From the above transfer function, one obtains (by inspection) the following vector differential equation input-output model:

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} + \frac{1}{RC} \begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} + \frac{1}{LC} \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2
\end{bmatrix} + \begin{bmatrix}
0 & \frac{1}{RC} \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2
\end{bmatrix} + \begin{bmatrix}
\frac{1}{T} & 0 \\
0 & -\frac{1}{T}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\] (7.66)

which relates the outputs \(y_1\) and \(y_2\) to the inputs \(u_1\) and \(u_2\).

### 7.7 Natural Modes, Exponential Stability, Eigenvalues, Eigenvectors, Matrix Exponential, And Modal Equations

As discussed earlier, every system has natural tendencies or natural modes. Each mode corresponds to a system pole (e.g. real exponential mode) or complex poles (e.g. exponential sinusoidal mode). As for SISO systems, a MIMO LTI system with state space representation \((A, B, C, D)\) is said to be exponentially stable if all of the eigenvalues of \(A\) (i.e. system poles) lie in the left half plane (i.e. have negative real parts).

The modes of an LTI system are best examined by applying standard tools from linear algebra, namely eigenvalues and eigenvectors, to the system matrix \(A\).

**Eigenvalues and Eigenvectors.** The eigenvalues \(\{s_i\}_{i=1}^n\) of a square matrix \(A \in \mathbb{R}^{n \times n}\) are the \(n\) roots of the characteristic equation

\[
\phi(s) \overset{\text{def}}{=} \det(sI - A) = 0.
\] (7.67)
Since [44]

\[(sI - A)^{-1} = \left[ \frac{\text{cof}(sI - A)}{\det(sI - A)} \right]^T, \tag{7.68}\]

it follows that the \(\{s_i\}_{i=1}^n\) are precisely the \(n\) system poles. The right eigenvectors \(v_i\) corresponding to the eigenvalues \(s_i\) of \(A\) are found by solving

\[(s_i I - A)v_i = 0 \tag{7.69}\]

for a nonzero \(v_i\) for each \(i = 1, 2, \ldots, n\). The \(s_i\) and \(v_i\) satisfy the matrix equation

\[AV = VA \tag{7.70}\]

where

\[V \overset{\text{def}}{=} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \quad \Lambda \overset{\text{def}}{=} \begin{bmatrix} s_1 & \cdots & \cdot & \cdots & s_n \end{bmatrix}. \tag{7.71}\]

In what follows, it will be assumed that the matrix \(A\) has \(n\) linearly independent right eigenvectors \(v_i\). In such a case, the \(n \times n\) matrix \(V\) is invertible. It is convenient to define its inverse \(W\) as follows:

\[W \overset{\text{def}}{=} \begin{bmatrix} -w_1^H & - & \cdots & - \\
- & - & \cdots & - \\
\vdots & \vdots & \ddots & \vdots \\
- & - & \cdots & - \end{bmatrix} = V^{-1} \tag{7.72}\]

where \(w_i^H\) denotes the \(i\)th row of \(W\) and \((\cdot)^H = (\cdot)^T\) denotes a conjugate-transpose operation [44]. Since \(V\) is invertible, it follows that \(WA = \Lambda W\) and that the \(w_i\) satisfy

\[w_i^H(s_i I - A) = 0 \tag{7.73}\]

for each \(i = 1, 2, \ldots, n\). Because of this, the \(w_i\) are called left eigenvectors of \(A\) corresponding to the eigenvalues \(s_i\).

**Dyadic Decomposition, Matrix Exponential.** Given the above, it follows that [44]

\[A = V\Lambda W = \sum_{i=1}^n s_i v_i w_i^H. \tag{7.74}\]
This shows that $A$ may be written as a linear combination of $n$ rank one matrices. The sum in equation (7.74) is referred to as a dyadic decomposition for $A$. From this, it follows that [44]

$$e^{At} = V e^{A} W = \sum_{i=1}^{n} e^{s_i t} v_i w_i^H$$

(7.75)

and

$$(sI - A)^{-1} = V(sI - \Lambda)^{-1} W = \sum_{i=1}^{n} \frac{1}{s - s_i} v_i w_i^H$$

(7.76)

(7.77)

Modal Relationships. Substituting these relationships into equations (7.26), (7.27), (7.28), (7.30), (7.31), (7.32) yields the following modal relationships:

$$X(s) = \sum_{i=1}^{n} \left[ w_i^H x_o + w_i^H B U(s) \right] \frac{1}{s - s_i} v_i$$

(7.78)

$$Y(s) = \sum_{i=1}^{n} \left[ w_i^H x_o + w_i^H B U(s) \right] \frac{1}{s - s_i} C v_i + DU(s)$$

(7.79)

$$H(s) = \sum_{i=1}^{n} \frac{C v_i w_i^H B}{s - s_i} + D$$

(7.80)

$$h(t) = \sum_{i=1}^{n} C v_i w_i^H B e^{s_i t} 1(t) + D \delta(t)$$

(7.81)

$$x(t) = \sum_{i=1}^{n} \left[ e^{s_i t} w_i^H x_o + \int_0^t e^{s_i (t-\tau)} w_i^H B u(\tau) d\tau \right] v_i$$

(7.82)

$$y(t) = \sum_{i=1}^{n} \left[ e^{s_i t} w_i^H x_o + \int_0^t e^{s_i (t-\tau)} w_i^H B u(\tau) d\tau \right] C v_i + Du(t)$$

(7.83)

These relationships are very useful for understanding the manner in which the natural modes of an LTI system impact the input-output and internal behavior of the system in the presence of initial conditions $x_o$ and inputs $u(\cdot)$.

7.8 Modal Analysis

The above modal relationships are very useful for understanding the natural tendencies of an LTI system. To examine the natural modes of the system, one sets the input $u$ to zero and examine the
state \(x(t)\). Doing so yields,

\[
x(t) = \sum_{i=1}^{n} [w_i^H x_o] e^{s_i t} v_i
\]

(7.84)

Next, it is essential to realize that \(W^V = I\) implies that the \(v_j\) and \(w_i\) are mutually orthonormal; i.e.

\[
w_i^H v_j \stackrel{\text{def}}{=} \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}
\]

(7.85)

With this, it follows that

\[
x_o = v_i \quad \implies \quad x(t) = e^{s_i t} v_i.
\]

(7.86)

This is the fundamental relationship for examining and understanding the natural modes of an LTI system. For a complex mode \(s_i = \sigma_i + j \omega_i\), it follows from equation (7.86) that

\[
x_o = a \Re\{v_i\} + b \Im\{v_i\} \quad \implies \quad x(t) = a \Re\{e^{s_i t} v_i\} + b \Im\{e^{s_i t} v_i\},
\]

(7.87)

### 7.9 Transmission Zeros And Energy Absorption

Just as some systems “give off” energy at certain frequencies - in the form of natural modes, some systems “absorb” certain frequencies. This motivates the concept of a zero. A SISO transfer function has a zero at \(s = s_0\) if \(H(s_0) = 0\). Analogously, a MIMO transfer function matrix \(H\) has a zero at \(s = s_0\) if \(H(s_0)\) loses rank. While this definition represents a direct extension of the classical SISO definition, we now give a definition for MIMO systems - a definition which captures our energy absorption notion of zeros.

Transmission Zeros. A dynamical system with input \(u\), state \(x\), and output \(y\), is said to have a transmission zero at \(s = s_0\) if there exists a pair of vectors \((u_0, x_0)\), such that \([x_0^T \quad u_0^T]^T \neq 0\) and such that

\[
u(t) = u_0 e^{s_0 t} \quad x(0) = x_0 \quad \implies \quad x(t) = e^{s_0 t} x_0 \quad y(t) = 0
\]

(7.88)
for all $t \geq 0$. This definition shows that the concept of a transmission zero is an energy absorption concept. The vectors $u_o$ and $x_o$ are said to define the direction of the transmission zero. The vector $u_o$ defines the input direction. The vector $x_o$ defines the state direction.

It can be shown that a MIMO LTI system with state space representation $(A, B, C, D)$ has a transmission zero at $s = s_o$ if and only if there exists $[x_o^T \quad u_o^T]^T \neq 0$ such that

$$
\begin{bmatrix}
    s_o I - A & -B \\
    C & -D
\end{bmatrix}
\begin{bmatrix}
    x_o \\
    u_o
\end{bmatrix} 
= 0.
$$

(7.89)

It is important to note that for SISO LTI systems, the definition given above for transmission zeros corresponds precisely the classical definition for zeros. For MIMO LTI systems, however, the transmission zeros of the system need not be zeros of any of the individual transfer functions within the system transfer function matrix! Examples which illustrate this point are readily constructed.

**Example 7.9.1 (Transmission Zeros And Transfer Function Matrix Entries.))**

Consider the LTI system

$$
A = \begin{bmatrix}
-1 & 0 & 0 \\
1 & -3 & 0 \\
0 & 0 & -2
\end{bmatrix} \quad B = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
$$

(7.90)

$$
C = \begin{bmatrix}
1 & -3 & 1 \\
5 & 0 & k
\end{bmatrix} \quad D = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
$$

(7.91)

This system has a transfer function matrix

$$
H(s) = \begin{bmatrix}
\frac{s}{(s+1)(s+3)} & \frac{1}{s+2} \\
\frac{5}{s+1} & \frac{k}{s+2}
\end{bmatrix}.
$$

(7.92)

It can be shown that for $k = 10$, the system has a transmission zero at $s_o = 3$ with input direction $u_o = [24 \quad -15]^T$ and state direction $x_o = [6 \quad 1 \quad -3]^T$. We note that for $k = 10$, the transmission zero at $s_o = 3$ is NOT a zero of any of the individual transfer function entries within the transfer function matrix $H(\cdot)$. 

$\blacksquare$
If \( x_o = 0 \), then we say that the system has a transmission zero at every \( s_o \) in the \( s \)-plane. If \( u_o = 0 \), then \( s_o \) is a system pole. If \( s_o \) is not a system pole, then \( s_o I - A \) is invertible and it follows that
\[
x_o = (s_o I - A)^{-1} Bu_o
\]
and
\[
H(s_o) u_o = 0
\tag{7.93}
\]
where \( H(s) \) is the system transfer function matrix. If the number of outputs \( p \) is equal to the number of inputs \( m \), then we say that the system is a square system. In such a case, the matrix \( C \) is square and the transmission zeros may be found by solving for the roots of the following polynomial:
\[
\det \begin{bmatrix} s_o I - A & -B \\ C & D \end{bmatrix} = \det(s_o I - A) \det H(s_o) = 0.
\tag{7.94}
\]

\[
(7.95)
\]

\[\blacksquare\]
Chapter 8

Propagation Of Signal Uncertainty Through An LTI System: Covariance Analysis

Many of the signals encountered in science and engineering contain “random” variations. Such variations are often referred to by the generic term: “noise.” Noise is typically modelled using concepts from probability theory and the theory of stochastic processes [24], [37]. Because of its widespread applicability, it is useful to examine the response of an LTI system to noise under uncertain initial conditions.

Consider the MIMO LTI system whose state space representation is given by

\[
\begin{align*}
\dot{x} &= Ax + L\xi; \quad x(t_0) = x_0 \\
y &= Cx + Du
\end{align*}
\]

(8.1)

(8.2)

where the sources of uncertainty are the initial condition vector \(x_0\) and the random input vector \(\xi(\cdot)\). This uncertainty is characterized by first and second order statistics as follows.

**Initial Condition, \(x_0\):**

\[
\begin{align*}
m_o &\overset{\text{def}}{=} E[x_0] \\
\Sigma_o &\overset{\text{def}}{=} E[(x_0 - m_o)(x_0 - m_o)^T]
\end{align*}
\]

(8.3)

(8.4)

where \(m_o \in \mathbb{R}^n, \Sigma_o \in \mathbb{R}^{n \times n}\);
Random Input, $\xi(\cdot)$:

\[ m_\xi(t) \overset{\text{def}}{=} E[\xi(t)] \tag{8.5} \]
\[ \Sigma_\xi(t) \overset{\text{def}}{=} E[(\xi - m_\xi)(\xi - m_\xi)^T] = \Xi \delta(t) \tag{8.6} \]
\[ x_o \quad \text{and} \quad \xi(\cdot) \quad \text{are uncorrelated} \tag{8.7} \]

where $\Xi = \Xi^T \geq 0$ is a symmetric positive semidefinite matrix [44].

State Mean. Given the above, it follows that the state mean $m_x(t) = E[x(t)]$ satisfies the relationship:

\[ m_x(t) = e^{A(t-t_o)}m_o + \int_{t_o}^{t} e^{A(t-\tau)}Lm_\xi(\tau)d\tau. \tag{8.8} \]

State and Output Autocovariance Matrices. Moreover, the autocovariance matrices $\Sigma_x(\cdot)$ and $\Sigma_y(\cdot)$ satisfy the following relationships [28, pg. 101]:

\[ \dot{\Sigma}_x(t) = A\Sigma_x(t) + \Sigma_x(t)A^T + L\Xi L^T; \quad \Sigma_x(t_o) = \Sigma_o \tag{8.9} \]
\[ \Sigma_x(t) = e^{A(t-t_o)}\Sigma_o e^{A^T(t-t_o)} + \int_{t_o}^{t} e^{A(t-\tau)}L\Xi L^T e^{A^T(t-\tau)}d\tau \tag{8.10} \]
\[ \Sigma_y(t) = C\Sigma_x(t)C^T + CD\Xi DT \tag{8.11} \]

Equation (8.9) is called a Lyapunov matrix differential equation and it shows that such equations occur naturally in studying the propagation of uncertainty through an LTI system.

Steady State Autocovariance Analysis. If $A$ is exponentially stable, then it follows that the limit

\[ \Sigma_x \overset{\text{def}}{=} \lim_{t \to \infty} \Sigma_x(t) \tag{8.12} \]

exists and satisfies the following steady state Lyapunov matrix algebraic equation:

\[ A\Sigma_x + \Sigma_x A^T + L\Xi L^T = 0. \tag{8.13} \]
Moreover,

\[ \Sigma_x = \int_0^\infty e^{A\tau} L Z L^T e^{A^T \tau} d\tau. \]  

(8.14)
Chapter 9

An Introduction To Feedback Control Systems: The Big Picture

9.1 Motivation

Many systems exhibit properties which may be undesirable from the perspective of a user. Because of this, engineers use feedback in an attempt to alter the undesirable properties. The following are specific reasons for using feedback:

- presence of undesirable natural modes (e.g. sluggish behavior, lightly damped oscillations, instabilities, undesirable high frequency dynamics, difficult nonlinear behaviors, etc.);
- presence of uncertainty (e.g. unknown or slowly varying parameters and dynamics, component degradation, unknown disturbances, etc.).

Any one of these reasons may make it impossible for a user to control the system without the assistance of some automatic feedback control system.

9.2 Standard Negative Feedback System

Because of the above issues, we propose the standard negative feedback system shown in Figure 9.1. In this figure, $P$ is called the plant and represents the physical system to be controlled (e.g. car system, airplane, etc.), $y$ represents the system output (e.g. actual speed), $r$ is called a reference command and represents the desired system output (e.g. desired speed), $e$ is an error signal to be
processed by a compensator $K$ (to be designed), $u$ is the control signal generated by $K$ (e.g. fuel flow to engine, elevator deflection, etc.), $d_i$ is a disturbance at the plant input (e.g. wind gust, etc.), $d_o$ is a disturbance at the plant output (e.g. wind gust, etc.), and $n$ represents sensor noise (e.g. measurement noise).

![Standard Negative Feedback System](image)

**Figure 9.1: Standard Negative Feedback System**

### 9.3 The Big Picture: A Small Tracking Error

If one defines the tracking error $e_T$ as follows:

$$e_T(t) \overset{\text{def}}{=} r(t) - y(t)$$  \hspace{1cm} (9.1)

then the goal becomes to design the compensator $K$ on the basis of a nominal plant model $P_o$ for $P$ such that the nominal closed loop system in Figure 9.2 ($P_o$ with $K$)

- is stable;
- exhibits a “small” tracking error for admissible reference commands, disturbances, and sensor noise; i.e. exhibits good nominal command following, disturbance rejection, and noise attenuation;
- is sufficiently robust to modeling errors between the nominal plant model $P_o$ and the actual plant $P$; i.e. the nominal feedback system in Figure 9.2 should be able to tolerate realistic
modeling uncertainty to the point that the above nominal closed loop properties are believed to hold for the actual closed loop system in Figure 9.1 (P with K).

The word “nominal” is intended to emphasize the fact that the nominal plant model \( P_o \) differs from the actual plant \( P \).

### 9.4 LTI Closed Loop Calculations

Assuming that \( P \) and \( K \) represent the transfer function matrices of LTI systems, then one can readily obtain a (zero initial condition) s-domain relationship between the Laplace transforms of the output \( y \) and that of the exogenous signals \( (r, d_i, d_o, n) \). This is done by “walking around” the feedback loop in Figure 9.1 - assuming zero initial conditions and exploiting linearity - as follows:

\[
y = y_p + d_o \tag{9.2}
\]

\[
y = Pu_p + d_o \tag{9.3}
\]

\[
y = P(u + d_i) + d_o \tag{9.4}
\]

\[
y = PK e + Pd_i + d_o \tag{9.5}
\]

\[
y = PK(r - y - n) + Pd_i + d_o \tag{9.6}
\]

\[
y = PKr - PKy + Pd_i + d_o - PKn \tag{9.7}
\]
or

\[ y = [I + PK]^{-1} PKr + [I + PK]^{-1} Pd_i + [I + PK]^{-1} d_o - [I + PK]^{-1} PKn \]  

(9.8)

If one defines the \textit{sensitivity} transfer function matrix as

\[ S \overset{\text{def}}{=} [I + PK]^{-1} \]  

(9.9)

and the \textit{complementary sensitivity} transfer function matrix as

\[ T \overset{\text{def}}{=} I - S = [I + PK]^{-1} PK \]  

(9.10)

then the above (zero initial condition) \(s\)-domain relationship for \( y \) becomes

\[ y = Tr + SPd_i + Sd_o - Tn. \]  

(9.11)

Given this, the expression for the tracking error becomes

\[ e_T = Sr - SPd_i - Sd_o + Tn. \]  

(9.12)

This important relationship loosely suggests that

- \( S \) should be small where \( r, Pd_i, \) and \( d_o \) have their frequency content, and that

- \( T \) should be small where \( n \) has its frequency content.

Since \((r, d_i, d_o)\) are typically low frequency signals and \( n \) is typically a high frequency signal, the above suggests that we typically want

- \( S \) small at low frequencies for good low frequency command following and disturbance rejection, and

- \( T \) small at high frequencies for good high frequency noise attenuation.

Given that the design \( K \) is to be based on the nominal model \( P_o \) for \( P \), and perhaps partial knowledge of the modeling errors which occur in representing \( P \) by \( P_o \), it follows that we want
the nominal sensitivity transfer function matrix and the nominal complementary transfer function matrix

\[ S_o \overset{\text{def}}{=} [I + P_o K]^{-1} \] (9.13)

\[ T_o \overset{\text{def}}{=} I - S_o = [I + P_o K]^{-1} P_o K \] (9.14)

to satisfy the above closed loop properties.

Methods for analyzing the feedback system in Figure 9.1 and for designing a compensator \( K \) are presented in many of the references. The theory for designing LTI compensators on the basis of LTI models is well developed. SISO feedback analysis and design methods (e.g. Root Locus Method, PID Controllers, Bode Plots, Nyquist Plots, Lead-lag Design, etc.) are addressed in [4], [12], [15], [19], [32], [38], [39]. MIMO control analysis and design methods are addressed in [28], [29], [50], [51]. State-of-the-art analysis and design support software is discussed in [11], [21], [30].

### 9.5 Stabilization

**Example 9.5.1 (Stabilization: Unstable Plant, Minimum Bandwidth Requirement.)**

Consider the feedback system in Figure 9.1 with nominal plant model

\[ P_o = \frac{g}{s - p} \quad g, p > 0 \] (9.15)

and constant gain compensator

\[ K = k. \] (9.16)

Such a controller is called a proportional controller. The above plant is unstable. The purpose of this example is to illustrate how feedback may be used to stabilize an unstable system.
The nominal open loop transfer function is

\[ L_o = \frac{k}{s - p} \]

(9.17)

Given this, we say that the nominal feedback system has a nominal open loop pole at \( s = p \). It also has a nominal open loop zero at \( \infty \). With this method of “accounting” - whereby finite and infinite roots are both counted, the number of poles is always equal to the number of zeros. For this system, a nominal \( s \)-domain zero initial condition relationship between the tracking error \( e_T \) and the exogenous signals \( (r, d_i, d_o, n) \) is as follows:

\[
e_T = S_o r - S_o P_o d_i - S_o d_o + T_o n
\]

(9.18)

\[
= \left[ \frac{1}{1 + P_o K} \right] r + \left[ \frac{-P_o}{1 + P_o K} \right] d_i + \left[ \frac{-1}{1 + P_o K} \right] d_o - \left[ \frac{P_o K}{1 + P_o K} \right] n
\]

(9.19)

\[
= \left[ \frac{s - p}{s - p + kg} \right] r + \left[ \frac{-1}{s - p + kg} \right] d_i + \left[ \frac{-(s - p)}{s - p + kg} \right] d_o + \left[ \frac{kg}{s - p + kg} \right] n.
\]

(9.20)

or in the time domain:

\[
\dot{e}_T + (kg - p)e_T = \dot{r} - pr - d_i - (\dot{d}_o - pd_o) + kgn.
\]

(9.21)

Given the above, the nominal closed loop characteristic equation becomes

\[ \phi_{d_s}(s) = s - p + kg \]

(9.22)

and hence the nominal closed loop system has a nominal closed loop pole at

\[ s = p - kg. \]

(9.23)

This implies that the nominal closed loop system is unstable for \( k < \frac{p}{g} \). It also implies that for \( k = \frac{p}{g} \), the nominal closed loop system has a closed loop pole at \( s = 0 \). Finally, and most importantly, we note that the

nominal closed loop system is stable if and only if \( k > \frac{p}{g} \).
This illustrates a fundamental idea, namely that a minimum gain (or bandwidth) is required to stabilize an unstable system. Moreover, the minimum gain (bandwidth) required is proportional to the instability (right half plane pole location). If you don’t believe this, try to balance a broom in your palm and then try to balance a toothpick. Why is the broom no problem to balance (over short periods of time) while the toothpick is impossible to balance? The answer is because the toothpick falls too fast for you to stabilize it. Put differently, you don’t have enough gain or bandwidth to stabilize it.

Now suppose that \( p = g = 1 \) and \( k = 10 \). Let’s examine what typical open loop and closed loop frequency responses look like. The open loop frequency response has been plotted in Figure 9.3. The sensitivity frequency response has been plotted in Figure 9.4. The complementary sensitivity frequency response has been plotted in Figure 9.5. The input disturbance to tracking error frequency response has been plotted in Figure 9.6. The reference to control frequency response has been plotted in Figure 9.7.

Figure 9.3: Open Loop Frequency Response: \( L = \frac{\text{\,10}}{s-1} \) (p = g = 1, k = 10)
Figure 9.4: Sensitivity Frequency Response: $S = \frac{s-p}{s-p+k_g}$ ($p = g = 1, k = 10$)

Figure 9.5: Complementary Sensitivity Frequency Response: $T = \frac{k_g}{s-p+k_g}$ ($p = g = 1, k = 10$)

While this example implies that the nominal closed loop system will be stable for all gains $k > \frac{p}{g}$, one cannot increase the gain (bandwidth) arbitrarily in a real feedback loop without exciting high frequency dynamics and risking oscillatory behavior and instability. This is true even if the plant is stable.

### 9.6 High Frequency Dynamics

**Example 9.6.1 (Effects of High Frequency Dynamics, Gain (Bandwidth) Restrictions.)**

Consider the feedback system in Figure 9.1 with nominal plant model

$$P_o = \frac{ab}{(s+a)(s+b)} \quad b > a > 0$$  \hspace{1cm} (9.24)
Figure 9.6: Input Disturbance to Tracking Error Frequency Response: \( T_{d,e_t} = \frac{-1}{s+9} \) \((p = g = 1, k = 10)\)

Figure 9.7: Reference To Control Frequency Response: \( T_{ru} = \frac{k(s-p)}{s-p+kg} \) \((p = g = 1, k = 10)\)

and constant gain compensator

\[ K = k. \]  \hspace{1cm} (9.25)

The nominal open loop transfer function is

\[ L_o^{\text{def}} = P_oK = k \frac{ab}{(s+a)(s+b)}. \]  \hspace{1cm} (9.26)

Given this, we say that the nominal feedback system has nominal open loop poles at \( s = -a \) and \( s = -b \). It also has two nominal open loop zeros at \( \infty \). For this system, a nominal s-domain zero initial condition relationship between the tracking error \( e_T \) and the exogenous signals \((r, d_i, d_o, n)\) is as follows:

\[ e_T = S_or - S_oP_od_i - S_od_o + T_on \]  \hspace{1cm} (9.27)
\[
\begin{align*}
\dot{e}_T + (a+b)\dot{e}_T + (1+k)abe_T &= \ddot{r} + (a+b)\ddot{r} + ab\dot{r} - \dot{d}_i - (\ddot{d}_o + (a+b)d_o + abd_o) + kab. & (9.30)
\end{align*}
\]

Given the above, the nominal closed loop characteristic equation becomes
\[
\phi_{CL_0}(s) = s^2 + (a+b)s + ab + kab
\]  
(9.31)

and the nominal closed loop poles are
\[
\begin{align*}
s_1 &= -\frac{(a+b)}{2} + \frac{\sqrt{(a+b)^2 - 4(1+k)ab}}{2} & (9.32) \\
s_2 &= -\frac{(a+b)}{2} - \frac{\sqrt{(a+b)^2 - 4(1+k)ab}}{2} & (9.33)
\end{align*}
\]

We now examine how the nominal closed loop poles depend on the design parameter \(k\). A plot in the \(s\)-plane which illustrates the variation of the closed loop poles as \(k\) is varied between 0 and \(+\infty\) is called a root locus [32].

It is well known that the nature of the roots are determined by the discriminant of the quadratic characteristic equation. Noting that the discriminant of the characteristic equation is zero for
\[
k = k_c \defeq \frac{(a+b)^2 - 4ab}{4ab},
\]
we examine the dependence of the closed loop poles on \(k\).

Overdamped Nominal Closed Loop System: \(k \in [0,k_c]\). For \(k = 0\), the nominal closed loop poles are at \(s = -a\) and \(s = -b\); i.e. they are precisely the nominal open loop poles. As \(k\) is increased, the fast pole at \(s = -b\) moves toward the right and the slow pole at \(s = -a\) moves toward the left.
For $k \in [0, k_c)$, the nominal closed loop system is exponentially stable and overdamped.

Critically Damped Nominal Closed Loop System: $k = k_c$. For $k = k_c$, the two nominal closed loop poles meet at $s = -(a + b)$ and the nominal closed loop system is exponentially stable and critically damped.

Underdamped Nominal Closed Loop System: $k \in (k_c, \infty)$. For $k > k_c$, the two nominal closed loop poles separate, become complex, and the nominal closed loop system becomes underdamped. Specifically,

$$s_1 = -\frac{a + b}{2} + j \frac{\sqrt{4(1 + k)ab - (a + b)^2}}{2} \tag{9.35}$$

moves upward along the line $s = -\frac{a + b}{2}$ and

$$s_2 = -\frac{a + b}{2} - j \frac{\sqrt{4(1 + k)ab - (a + b)^2}}{2} \tag{9.36}$$
moves downward. As $k$ is increased, the nominal closed loop poles move further and further away from the real axis along the line $s = -\frac{a + b}{2}$. As such, their damping factor decreases and the nominal closed loop system becomes less and less damped.

The above closed loop modal analysis may be visualized as shown in Figure 9.8. In this figure, the nominal closed loop poles are plotted in the $s$-plane for $k \geq 0$. The plot is called a root locus for the nominal feedback system. The plot confirms the above nominal closed loop analysis.

The above analysis shows that the closed loop system is exponentially stable for all $k \geq 0$. Next, we examine the steady state tracking error to step reference commands. Since the nominal closed loop system is exponentially stable, the steady state tracking error resulting from a constant reference command is a constant (see section on steady state analysis). Specifically, the steady state tracking error resulting from a unit step reference command is given by the dc gain of the sensitivity transfer
function as follows:

\[ e_{T_{ss}} = S(0) = \frac{1}{1 + k}. \]  \hspace{1cm} (9.37)

This shows that as \( k \) (and hence the bandwidth) is increased, the nominal tracking error can be made arbitrarily small. While this follows from the above nominal (mathematical) model, physical intuition and practical experience tell us that \( k \) (and the bandwidth) cannot be increased arbitrarily in practice. The reason for this is the sure presence of (uncertain) high frequency dynamics which have been neglected in our nominal model \( P_o \). This is now shown.

Suppose that the actual plant \( P \) is given by

\[ P(s) = P_o(s) \frac{c}{s + c}. \]  \hspace{1cm} (9.38)

where \( c > b > a > 0 \). For this \( P \), the actual closed loop characteristic equation becomes:

\[ \Phi_{CL}(s) = (s + a)(s + b)(s_c) + kabc = s^3 + (a + b + c)s^2 + (ab + bc + ac)s + abc + kabc. \]  \hspace{1cm} (9.39)

The main point to be made in this example is that if \( k \) (and hence the bandwidth) is made too large, the actual closed loop system will go unstable. How do we show this? The statement implies that
if $k$ is chosen large enough, then there will be at least one closed loop pole on the imaginary axis. Given this, lets determine the critical value of $k$ for which this happens.

Determining Imaginary Crossovers. Substituting $s = j\omega$ into equation (9.39), yields the following:

$$
\Phi_{CL}(j\omega) = (j\omega)^3 + (a + b + c)(j\omega)^2 + (ab + bc + ac)(j\omega) + abc + kabc.
$$

(9.40)

Setting this equal to zero yields

$$
\Phi_{CL}(j\omega) = -j(\omega)^3 - (a + b + c)\omega^2 + (ab + bc + ac)j\omega + abc + kabc = 0.
$$

(9.41)

This single complex equation in the two unknowns $k$ and $\omega$ yields two real equations which are obtained by equating the real and imaginary parts of equation (9.41) to zero as follows:

$$
-(a + b + c)\omega^2 + abc + kabc = 0,
$$

(9.42)

$$
-(\omega)^3 + (ab + bc + ac)\omega = 0.
$$

(9.43)

Equation (9.43) yields two solutions

$$
\omega = 0,
$$

(9.44)

$$
\omega = \sqrt{ab + bc + ac}.
$$

(9.45)

From equation (9.42), one obtains

$$
k = \frac{(a + b + c)\omega^2 - abc}{abc}
$$

(9.46)

which in turn yields two values of $k$ as follows:

$$
k = -1 \text{ corresponds to } \omega = 0,
$$

(9.47)

and

$$
k = k_i \equiv 2 + \left(\frac{b}{a} + \frac{a}{b}\right) + \left(\frac{c}{b} + \frac{b}{c}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) \text{ corresponds to } \omega = \omega_i \equiv \sqrt{ab + bc + ac}.
$$

(9.48)
The negative value of $k$ is irrelevant since we are only considering $k \geq 0$. The pair $(k_i, \omega_i)$ are said to define an imaginary crossover since for $k = k_i$ there are two closed loop poles on the imaginary axis, namely at $s = \pm j\omega_i$.

If one plots the actual closed loop poles in the $s$-plane for $k \geq 0$, one obtains the plot in Figure 9.9. This figure is a root locus for the actual feedback system and confirms the above analysis.

![Root Locus For Actual Feedback System](image)

Figure 9.9: Root Locus For Actual Feedback System: $L = \frac{k}{(s+a)(s+b)(s+c)}$ ($a = 1, b = 2, c = 10$)

In summary, this example demonstrates that in practice one, in general, cannot arbitrarily increase the gain (bandwidth) within a feedback loop because there will always be (uncertain) high frequency dynamics which will cause problems (e.g. undesirable oscillations or perhaps closed loop instability). Such dynamics might include, actuator dynamics, flexible modes, etc.

### 9.7 Internal Model Principle

In many applications it is desirable to track a specific reference command $r$ or reject a specific disturbance $d$ in the steady state. While it has been shown that a large gain in the feedback
loop may result in a small steady state tracking error $e_{T_{ss}}$, it has also been seen that in real feedback systems gains cannot be increased arbitrarily without risking severe degradation in system performance (e.g. oscillatory behavior or instability). Rather than pursuing a large gain strategy, we now propose a strategy based on the so-called internal model principle. To explain this principle we use an example which everyone can relate to.

Suppose that an individual (at a carnival) is trying to shoot a moving (metallic) duck whose motion appears to be almost periodic. Given this, it follows that the determined shooter must try to develop an “internal model” for the duck’s motion. Analogously, the internal model principle states that if a feedback system is to track a specific reference command or reject a specific disturbance, then the feedback loop must possess an internal model for the reference command and disturbance. This principle is arguably a very intuitive and compelling one. The following example illustrates how the principle may be exploited in practice.

**Example 9.7.1 (Internal Model Principle, Command Following, Disturbance Rejection.)**

*In this example, we examine the ability or inability of the feedback system Figure 9.1 to track step commands $r$ and reject step disturbances $d_i$ and $d_o$ in the steady state. It will be assumed that $n = 0$ (no sensor noise) so that the tracking error $e_T$ and the error signal $e$ are identical. Given the above, it follows that*

$$e_T = Sr - SPd_i - Sd_o. \tag{9.49}$$

*Assuming that the closed loop system is exponentially stable, it then follows that the steady state tracking error resulting from step commands and disturbances is given by*

$$e_{T_{ss}} = S(0)r_{ss} - S(0)P(0)d_{i_{ss}} - S(0)d_{o_{ss}} \tag{9.50}$$

*where $r_{ss}$, $d_{i_{ss}}$, $d_{o_{ss}}$ denote the steady state values of $r$, $d_i$, and $d_o$, respectively.*
Because step commands and disturbances are of interest, and since the Laplace transform of a unit step is an integrator $\frac{1}{s}$, it follows naturally - by the internal model principle - that we need an integrator somewhere within our feedback loop (Figure 9.1) in order to be able to track step commands and reject step disturbances in the steady state. But where exactly in the feedback loop should the integrator reside? Within the plant $P$? Within the compensator $K$? We now examine two cases to answer this question precisely.

Case 1: Internal Model Within Plant $P$. We begin by considering an unstable plant

$$P = \frac{1}{s(s - 1)}$$

with a compensator

$$K = 2(s + 0.5).$$

This compensator structure is referred to as a proportional plus derivative or PD controller. Because we are interested in step command following and disturbance rejection, it follows that the plant $P$ contains an internal model; i.e. an integrator.

The closed loop characteristic equation is

$$\Phi_{CL}(s) = s(s - 1) + 2(s + 0.5) = s^2 + s + 1.$$  \hspace{1cm} (9.53)

This implies that the closed loop poles are

$$s_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$  \hspace{1cm} (9.54)

and that the closed loop system is exponentially stable. Given this, it makes sense to discuss steady state step command following and disturbance rejection. Specifically, since

$$e_T = \left[ \frac{s(s - 1)}{s^2 + s + 1} \right] r + \left[ \frac{-1}{s^2 + s + 1} \right] d_i + \left[ \frac{-s(s - 1)}{s^2 + s + 1} \right] d_o,$$  \hspace{1cm} (9.55)

it follows that the steady state tracking error is given by

$$e_{Tss} = S(0)r_{ss} - S(0)P(0)d_{i_{ss}} - S(0)d_{o_{ss}} = [0] + [-d_{i_{ss}}] + [-0] = -d_{i_{ss}}.$$  \hspace{1cm} (9.56)
This simple steady state calculation shows that

- step reference commands \( r \) are followed in the steady state,
- step output disturbances \( d_o \) are rejected in the steady state,
- step input disturbances \( d_i \) are NOT rejected in the steady state.

Closer term-by-term examination shows that \( r \) is followed and \( d_o \) is rejected because of the zero at \( s = 0 \) in the sensitivity transfer function \( S \). This zero, however, is directly due to the internal model (integrator) in the plant \( P \). \( d_i \) is not rejected because the internal model is in \( P \) and not in \( K \). A consequence of this is that the transfer function from \( d_i \) to \( e_T \) \((-SP)\) does not contain a zero at \( s = 0 \).

Case 2: Internal Model Within Compensator \( K \). Now suppose that

\[
P = \frac{1}{s - 1} \tag{9.57}
\]

with a compensator

\[
K = 2 \left[ \frac{s + 0.5}{s} \right] . \tag{9.58}
\]

This compensator structure is referred to as a proportional plus integral or PI controller. In this case the compensator \( K \) contains an internal model; i.e. an integrator.

The closed loop characteristic equation, closed loop poles, and the exponential stability of the closed loop system remain unaltered. In this case, we have

\[
e_T = \left[ \frac{s(s - 1)}{s^2 + s + 1} \right] r + \left[ \frac{-s}{s^2 + s + 1} \right] d_i + \left[ \frac{-s(s - 1)}{s^2 + s + 1} \right] d_o , \tag{9.59}
\]

from which it follows that the steady state tracking error is given by

\[
e_T^{ss} = S(0)r^{ss} - S(0)P(0)d_i^{ss} - S(0)d_o^{ss} = [0] + [-0] + [-0] = 0 . \tag{9.60}
\]

This simple steady state calculation shows that

- step reference commands \( r \) are followed in the steady state,
• step output disturbances $d_o$ are rejected in the steady state,

• step input disturbances $d_i$ are rejected in the steady state.

Closer term-by-term examination shows that $r$ is followed and the pair $(d_i, d_o)$ are rejected in the steady state because each of the transfer functions which relates these to the tracking error $e_T$ contains a zero at $s = 0$. These zeros, however, are directly due to the internal model (integrator) in the compensator $K$.

Case 1 above illustrates that it is not sufficient for an internal model to be present within the feedback loop. Case 1 shows that where the internal model appears is very important. Specifically, the above calculations illustrate that the

Internal model must process the error signal
prior to the point of injection of the signal of interest.

Given this observation, we note that $r$ is followed in case 1 and case 2 because in both cases, the internal model processes the error signal prior to the point of injection of $r$. Similarly, $d_o$ is rejected in case 1 and case 2 because in both cases, the internal model processes the error signal prior to the point of injection of $d_o$. In contrast, $d_i$ is NOT rejected in case 1 because the error signal is NOT processed by the internal model prior to the point of injection of $d_i$. It is rejected in case 2 because the error signal is processed by the internal model prior to the point of injection of $d_i$.

The internal model principle is a very useful analysis tool. It is even more powerful as a design tool. If it is necessary to follow a sinusoid or reject a disturbance of frequency $\omega_o$, then an internal model $\frac{1}{s^2 + \omega_o^2}$ is required within the feedback loop. If a sinusoidal input disturbance $d_i$ of frequency $\omega_o$ must be rejected, then the internal model must reside within the compensator $K$. If a sinusoidal output disturbance $d_o$ of frequency $\omega_o$ must be rejected, the internal model may reside within the
plant $P$ or within the compensator $K$. By continuity, it follows that an internal model with stable lightly damped poles near $\pm j\omega_0$ would result in a small tracking error - provided, of course, that the closed loop system has been designed to be stable.

### 9.8 Feedforward Compensation

While the feedback system in Figure 9.1 is widely used and extremely convenient for conveying fundamental control principles, it is sometimes useful to consider a slightly different feedback system structure. This fact is addressed in the following example.

**Example 9.8.1 (Feedforward Compensation To Shape Response To Step Commands.)**

Consider the feedback system in Figure 9.1 with plant

$$P = \frac{1}{s - 1} \quad (9.61)$$

and PI controller

$$K = 2 \left[ \frac{s + 0.5}{s} \right]. \quad (9.62)$$

The closed loop characteristic equation is

$$\Phi_{CL}(s) = s(s - 1) + 2(s + 0.5) = s^2 + s + 1. \quad (9.63)$$

The closed loop poles are

$$s_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}. \quad (9.64)$$

From this, it follows that the closed loop system is exponentially stable, underdamped, with closed loop poles having a damping factor $\zeta = 0.5$. One might expect, based on formula (5.105) and Figure 5.5, that the response to a step reference command would look great with an overshoot less than 20%. This, however, should not be expected because the transfer function from $r$ to $y$

$$T_{ry}(s) = \frac{2(s + 0.5)}{s^2 + s + 1} \quad (9.65)$$
is not in standard second order form (see equation (5.55)). This is because of the numerator polynomial in equation (9.65). A consequence of this term is to provide additional derivative plus proportional action when a step is applied. The extra derivative action differentiates the step reference command to produce an impulse. The result is extra overshoot as exhibited in Figure 9.10.

![Graph showing Overshoot Due To Zero In PI Series Compensator](chart.png)

Figure 9.10: Closed Loop Step Response: Overshoot Due To Zero In Series Compensator

To correct this problem, we propose the slightly different feedback structure in Figure 9.11 with series compensator $K_1$ and feedforward compensator $K_2$.

Suppose that

$$K_1 = 2 \left[ \frac{s + 0.5}{s} \right].$$

(9.66)

and

$$K_2 = -K_1(\infty) = -2.$$  

(9.67)
In such a case the transfer function from \( r \) to \( y \) is as follows:

\[
T_{ry}(s) = \frac{1}{s^2 + s + 1}. \tag{9.68}
\]

This transfer function is in standard form with \( \zeta = 0.5 \) and \( \omega_n = (\text{see equation (5.55)}). \) As such, its step response exhibits the desired less than 20% overshoot characteristics (see Figure 9.12).

Figure 9.11: Negative Feedback System With Series and Feedforward Compensation

Figure 9.12: Feedforward Compensation Corrects Overshoot To Step Reference Command
9.9 Stability Robustness Margins

Example 9.9.1 (Stability Robustness Margins, Root Locus and Frequency Response Ideas.)

Consider the feedback system in Figure 9.1 with open loop transfer function

\[ L(s) = P(s)K(s) = 8k \frac{(s - 0.5)}{(s - 2)(s - 3)} \] (9.69)

where \( k \geq 0 \). The nominal value for \( k \) is 1. The purpose of this example is to examine fundamental robustness properties associated with the feedback loop.

Equation (9.69) shows that the feedback loop has two unstable open loop poles at \( s = 2, 3 \) and one right half plane (non-minimum phase [32]) zero at \( s = 0.5 \). We now examine how the closed loop poles depend on \( k \). The closed loop characteristic equation is given by:

\[ \phi_{CL}(s) = (s - 2)(s - 3) + 8k(s - 0.5) = s^2 + (8k - 5)s + 6 - 4k. \] (9.70)

The closed loop poles are given by:

\[ clp_1 = -\frac{8k - 5}{2} + \frac{\sqrt{(8k - 5)^2 - 4(6 - 4k)}}{2} \] (9.71)

\[ clp_2 = -\frac{8k - 5}{2} - \frac{\sqrt{(8k - 5)^2 - 4(6 - 4k)}}{2}. \] (9.72)

We begin our analysis by noting that the discriminant is zero when

\[ \text{discriminant} = (8k - 5)^2 - 4(6 - 4k) = 64k^2 - 64k + 1 = 0 \] (9.73)

or \( k_1 = 0.0159 \) and \( k_2 = 0.9841 \). The corresponding closed loop poles are found to be \( s_1 = 2.4365 \) and \( s_2 = -1.4365 \), respectively.

\( k \in [0, k_1] \). For \( k = 0 \), the closed loop poles are at \( s = -1 \) and \( s = -2 \) - the location of the open loop poles. As \( k \) is increased the rightmost pole \( clp_1 \) moves toward the left and the leftmost pole \( clp_2 \) moves toward the right. The two closed loop poles meet at \( s = s_1 = 2.4365 \) for \( k = k_1 = 0.0159 \).
\(k \in (k_1, k_2]\). As \(k\) is increased above \(k_1 = 0.0159\), the two poles separate and become complex. clp_1 moves upward and clp_2 moves downward. At \(k = k_{i_1} = 5/8 = 0.625\), the closed loop poles are both on the imaginary axis at \(s = \pm j \omega_{i_1} = \pm j \sqrt{3.5} = \pm j1.8708\). We say that there is an imaginary crossover at \((k_{i_1} = 5/8, \omega_{i_1} = 1.8708)\). As \(k\) is increased further, the closed loop poles meet at \(s = s_2 = -1.4365\) for \(k = k_2 = 0.9841\). It can be shown that for \(k \in (k_1, k_2]\), the closed loop poles lie on a circle with center at \(s = 0.5\) (open loop zero), passing through \((s_1, s_2, \pm j \omega_{i_1})\) with radius \(r = \sqrt{(0.5)^2 + \omega_{i_1}^2} = \sqrt{3.75}\). The equation of the circle is given by

\[
(\sigma + 5)^2 + \omega^2 = 3.75 \tag{9.74}
\]

where

\[
\sigma = -\frac{8k - 5}{2}, \quad \omega = \frac{\sqrt{(8k - 5)^2 - 4(6 - 4k)}}{2} \tag{9.75}
\]

\(k \in (k_2, \infty)\). As \(k\) is increased above \(k_2\) one pole (clp_2) moves toward the left and the other pole (clp_1) moves toward the right. For \(k = k_{i_2} = 1.5\), the closed loop poles are at \(s = 0\) and \(s = -7\). We thus have another imaginary crossover at \((k_{i_2} = 1.5, \omega_{i_2} = 0)\). As \(k\) is increased above \(k_{i_2}\) the leftmost pole (clp_2) moves deeper into the left half plane while the rightmost pole (clp_1) approaches the open loop zero at \(s = 0.5\).

Closed Loop Stability Summary. The above shows that the closed loop system is unstable for \(k \in [0, k_{i_1}]\) with two poles in the right half plane. It is marginally stable for \(k = k_{i_1} = 5/8\) with closed loop poles at \(s = \pm j \omega_{i_1} = \pm j1.8708\). For \(k \in (k_{i_1}, k_{i_2}]\), the closed loop system is stable with two poles in the left half plane. For \(k = k_{i_2} = 1.5\) the closed loop system is marginally stable with one pole at \(s = j \omega_{i_2} = 0\) and the other at \(s = -7\). Finally, for \(k \in (k_{i_2}, \infty)\) the closed loop system is unstable with one pole in the right half plane (clp_1) and the other in the left half plane (clp_2). The above analysis may be visualized by the root locus plot in Figure 9.13. Methods for systematically constructing root loci are contained in each of the classical SISO control references [12], [15], [19], [32], [38], [39].
Nominal Design. Now let's examine the robustness properties of the closed loop system corresponding to the nominal design parameter $k = k_0 = 1 \in (k_{i1}, k_{i2})$. For this nominal value, the closed loop system is stable with poles at $s = -1$ and $s = -2$. We now consider several fundamental questions.

Downward Gain Margin, $\downarrow \text{GM}$. How much can the nominal gain $k = k_0$ be decreased before the closed loop system goes unstable? This question is called the downward gain margin and is given by $\downarrow \text{GM} = \frac{k_{i1}}{k_0} = 5/8$. Downward gain margin is always less than unity for a closed loop stable feedback system. It should be noted that whenever there is an open loop right half plane pole, the downward gain margin will be greater than zero. This is because in such a case, some closed loop pole will migrate towards an open loop right half plane pole as the gain $k$ is decreased toward zero.

Upward Gain Margin, $\uparrow \text{GM}$. How much can the nominal gain $k = k_0$ be increased before the closed loop system goes unstable? The answer to this question is called the upward gain margin and is
given by $\Uparrow GM = \frac{k_2}{k_0} = 1.5$. Upward gain margin is always greater than unity for a closed loop stable feedback system. It should be noted that whenever there is an open loop right half plane zero, the upward gain margin will be finite; i.e. not infinite. This is because in such a case, some closed loop pole will migrate towards an open loop right half plane zero as the gain $k$ is increased towards $\infty$.

Gain Margins From Open Loop Frequency Response. The nominal open loop frequency response is plotted in Figure 9.14. From this plot one can read off the downward and upward gain margins. The downward gain margin of $\Uparrow GM = 5/8$ (−4.0824 db) may be read off at $\omega_i_1 = 1.8708$. The upward gain margin of $\Uparrow GM = \frac{k_2}{k_0} = 1.5$ (3.5218 db) may be read off at $\omega_i_2 = 0$. These frequencies $\omega_i_1, \omega_i_2$ correspond to imaginary crossovers on the root locus. Notice that both of these correspond to frequencies such that

$$\angle L(j\omega) = 180 \pm m360^\circ \quad m = 0, 1, \ldots$$  \hspace{1cm} (9.76)

These frequencies are called phase crossover frequencies. They correspond to imaginary crossovers on the root locus and can always be read from the open loop phase response. This discussion ties root locus, frequency response, and stability robustness ideas.

Phase Margins ($\Uparrow PM, \Uparrow PM_i$) From Open Loop Frequency Response. Just as one can examine the stability robustness of a stable closed loop feedback system to gain uncertainty, it is also natural to examine its robustness to phase uncertainty. This motivates the concept of phase margin. Given this, the downward phase margin $\Uparrow PM$ of a stable closed loop feedback system is defined to be the amount of phase lag $e^{-j\Uparrow PM}$ which may be inserted into the feedback loop before the closed loop system goes unstable. Similarly, the upward phase margin $\Uparrow PM$ of a stable closed loop feedback system is defined to be the amount of phase lead $e^{j\Uparrow PM}$ which may be inserted into the feedback loop before the closed loop system goes unstable. Because the closed loop system is on the verge of instability when it has closed loop poles on the imaginary axis, it follows that downward phase
margin is determined by the following formula:

\[ 1 + e^{-jPM} L(j\omega) = 0 \]  

\[ (9.77) \]

or the two formulae:

\[ |L(j\omega)| = 1 \]  

\[ (9.78) \]

\[ \downarrow PM = 180^\circ + \angle L(j\omega) \pm m360^\circ \quad m = 0, 1, \ldots \]  

\[ (9.79) \]

The equations imply that we should determine the frequency \( \omega_g \) at which \( |L(j\omega_g)| = 1 \). This frequency is called the gain crossover frequency and may be read off from the magnitude response. The phase margin is then found by using equation (9.79). Similarly, after one determines the gain crossover frequency the upward phase margin is found using the following formula:

\[ \uparrow PM = 180^\circ - \angle L(j\omega) \pm m360^\circ \quad m = 0, 1, \ldots \]  

\[ (9.80) \]

\[ L = \frac{8(s - 0.5)}{(s - 2)(s - 3)} \]

Figure 9.14: Open Loop Frequency Response: \( L = \frac{8(s - 0.5)}{(s - 2)(s - 3)} \)
From Figure 9.14, it follows that the gain crossover frequencies are $\omega_1 = 0.63 \text{ rad/sec}$ and $\omega_2 = 7 \text{ rad/sec}$. These frequencies yield the upward and downward phase margins

$$\uparrow PM = 180^\circ - \angle L(j\omega_1) = 180^\circ - 157.78^\circ = 22.22^\circ$$  \hspace{1cm} (9.81)

$$\downarrow PM = \angle L(j\omega_2) - 180^\circ = 234.94^\circ - 180^\circ = 54.94^\circ.$$  \hspace{1cm} (9.82)
Chapter 10

Controllability: Transferring The State Of A System

When can the natural tendencies of a system be altered via feedback? To help answer this fundamental question, it is useful to introduce the concept of controllability [8], [26]. A system is said to be controllable at \( t = t_o \) if for any initial state \( x(t_o) = x_o \) and desired final state \( x_f \), there exists a finite time \( t_f > t_o \) and a control \( u(\cdot) \) on \([t_o, t_f)\) which transfers the state of the system from \( x(t_o) = x_o \) to \( x(t_f) = x_f \). Otherwise the system is said to be uncontrollable at \( t_o \). In examining the controllability of an LTI system \((A, B, C, D)\), it is useful to define two matrices:

\[
C \overset{\text{def}}{=} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times m n} \tag{10.1}
\]

\[
G_c(t_1, t_2) \overset{\text{def}}{=} \int_{t_1}^{t_2} e^{A(t_1-\tau)} B B^H e^{A^H(t_1-\tau)} d\tau \in \mathbb{R}^{n \times n}. \tag{10.2}
\]

The matrix \( C \) is called the system controllability matrix. The matrix \( G_c(t_1, t_2) \) is called the system controllability grammian. Given the above definitions, the LTI system \((A, B, C, D)\) is controllable (at any \( t_o \)) if and only if

\[
\text{rank } C = n. \tag{10.3}
\]

If \( C \) has rank less than \( n \), then we say that the system is uncontrollable. Alternately, \((A, B, C, D)\) is controllable (at any \( t_o \)) if and only if

\[
\text{rank } G_c(t_o, t_f) = n \tag{10.4}
\]
for some $t_f$. Another very useful controllability test is the Popov-Belevitch-Hautus modal controllability test [26]. An LTI system $(A, B, C, D)$ is controllable (at any $t_o$) if and only if there does not exist an eigenvalue $s_i$ with a corresponding left eigenvector $w_i^H$ which lies in the left null space of $B$; i.e.

$$w_i^H \left[ s_i I - A \quad -B \right] \neq 0^H \quad (10.5)$$

for all eigenvalues $s_i$ and associated left eigenvectors $w_i^H$. If

$$w_i^H \left[ s_i I - A \quad -B \right] = 0^H \quad (10.6)$$

for some $i$, then we say that the system is uncontrollable and, more specifically, that the mode $s_i$ is an uncontrollable mode. If $(A, B, C, D)$ is controllable, then $G_c(t_o, t_f)$ is invertible for any $t_f > t_o$. Moreover, a control which accomplishes the transfer of state from $x(t_o) = x_o$ to $x(t_f) = x_f$ is as follows [8]:

$$u(t) = B^H e^{A^H(t_o - t)} G_c(t_o, t_f)^{-1} \left[ e^{A(t_o - t_f)} x_f - x_o \right]. \quad (10.7)$$

It can be shown that this control accomplishes the state transfer by expending minimum energy [8].
Chapter 11

Full State Feedback: A Method For Altering The Natural Modes Of A System

In trying to alter the behavior of a system via feedback, it is natural to consider the case where the entire state of the system is available for feedback. This is sometimes referred to as the full information case [50]. Now consider the LTI system \((A, B, C, D)\). We begin by examining simple constant gain full state feedback control laws having the form

\[
u(t) = v(t) - Gx(t)
\]

(11.1)

where \(G \in \mathbb{R}^{m \times n}\) is called the control gain matrix and \(v(\cdot)\) can be thought of as a desired control. This control law may be visualized as shown in Figure 11.1.

Figure 11.1: Block Diagram Visualization of LTI System Under Constant Gain Full State Feedback
The state space representation for the closed loop system is as follows:

\[
\begin{align*}
\dot{x}(t) &= (A - BG)x(t) + Bv(t) \\
y &= (C - DG)x(t) + Dv(t).
\end{align*}
\]

The closed loop poles are the eigenvalues of the closed loop matrix: \( A - BG \). It can be shown that the control gain matrix \( G \) may be selected to place the closed loop poles at arbitrary locations in the s-plane (with complex poles occuring in conjugate pairs) if and only if the pair \((A, B)\) is controllable [8], [26]. Moreover, it can be shown that the original system \((A, B, C, D)\) is stabilizable via constant gain full state feedback if and only if all of the unstable poles of \( A \) are controllable. Many authors take this as the definition for a stabilizable LTI system \((A, B, C, D)\) [26]. Unstable modes which are not controllable are said to be \textit{unstabilizable}.

Sometimes the state \( x(\cdot) \) is not available for feedback and only the input \( u(\cdot) \) and the output \( y(\cdot) \) are available. In such a case, we say that we have \textit{partial information}. When this is the case important issues arise. These issues will be discussed subsequently.
Chapter 12

Observability: Reconstructing The State Of A System

It has been shown that knowledge of the state $x(\cdot)$ is very useful for constructing desirable control laws. What does one do if the state is not available for feedback? What if only the input and output are available?

Also, it is “intuitive” that in order to be able to modify the “undesirable behaviors” of a system via feedback, those “undesirable behaviors” need to be “visible” in the measurement being fed back. Given the above, it is natural to consider the following two questions:

- When can the state $x(\cdot)$ be reconstructed from knowledge of the input $u(\cdot)$ and the output $y(\cdot)$?

- When do the feedback measurements contain all of the “undesirable behaviors”? (e.g. When are the instabilities “visible”?)

To help answer these fundamental questions, it is useful to introduce the concept of observability [8], [26]. A system is said to be observable at $t_o$ if there exists a finite time $t_f > t_o$ such that the initial condition $x(t_o) = x_o$ can be uniquely determined given knowledge of the input $u(\text{cot})$ and the output $y(\cdot)$ on $[t_o, t_f]$. Otherwise the system is said to be unobservable at $t_o$. In examining the
observability of an LTI system \((A, B, C, D)\), it is useful to define two matrices:

\[
O \overset{\text{def}}{=} \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} \in \mathbb{R}^{n \times npn}
\]  
(12.1)

\[
G_o(t_1, t_2) \overset{\text{def}}{=} \int_{t_1}^{t_2} e^{A^H(t-t_1)} C^H C e^{A(t-t_1)} d\tau \in \mathbb{R}^{n \times n}.
\]  
(12.2)

The matrix \(O\) is called the system observability matrix. The matrix \(G_o(t_1, t_2)\) is called the system observability grammian. Given the above definitions, the LTI system \((A, B, C, D)\) is observable (at any \(t_0\)) if and only if

\[
\text{rank } O = n.
\]  
(12.3)

If \(O\) has rank less than \(n\), then we say that the system is unobservable. Alternately, \((A, B, C, D)\) is observable (at any \(t_0\)) if and only if

\[
\text{rank } G_o(t_0, t_f) = n
\]  
(12.4)

for some \(t_f\). Another very useful observability test is the Popov-Belevitch-Hautus modal observability test [26]. An LTI system \((A, B, C, D)\) is observable (at any \(t_0\)) if and only if there does not exist an eigenvalue \(s_i\) with a corresponding right eigenvector \(v_i\) which lies in the right null space of \(C\); i.e.

\[
\begin{bmatrix}
s_i I - A \\
C
\end{bmatrix} v_i \neq 0^H
\]  
(12.5)

for all eigenvalues \(s_i\) and associated right eigenvectors \(v_i\). If

\[
\begin{bmatrix}
s_i I - A \\
C
\end{bmatrix} v_i = 0^H
\]  
(12.6)

for some \(i\), then we say that the system is unobservable and, more specifically, that the mode \(s_i\) is an unobservable mode. If \((A, B, C, D)\) is observable, then \(G_o(t_0, t_f)\) is invertible for any \(t_f > t_0\).

Moreover, the initial condition \(x_0\) may be uniquely determined from knowledge of \(u(\cdot)\) and \(y(\cdot)\) on \([t_0, t_f]\) by using the relationship [8]

\[
x_0 = G_o(t_0, t_f)^{-1} \int_{t_0}^{t_f} e^{A^H(t-t_0)} C^H \tilde{y}(\tau) d\tau
\]  
(12.7)
where

\[ \bar{y}(t) \overset{\text{def}}{=} y(t) - \int_{t_s}^{t} e^{A(t-\alpha)} Bu(\alpha) d\alpha - Du(t). \]  

(12.8)

While the above suggests a method for “reconstructing” the initial state, the method can be very sensitive to modeling errors. A more robust (i.e. less sensitive) method which employs feedback is presented below.
Chapter 13

State Observers: State Reconstruction Or Estimation Via Feedback

Suppose that we have a plant whose state $x(\cdot)$ is unknown. We would like to construct $x(\cdot)$ given knowledge of the input $u(\cdot)$ and $y(\cdot)$. An observer or state estimator is a dynamical system whose purpose is to reconstruct the state $x(\cdot)$ from knowledge of the input $u(\cdot)$ and the output $y(\cdot)$ (see Figure 13.1).

![Figure 13.1: Visualization of State Observer (Estimator)](image)

We now present a well known observer feedback structure which is based on mimicking the original plant dynamics. It is assumed that the plant is an LTI system with state space representation $(A, B, C, D)$. If $\hat{x}$ denotes our “estimate” for $x$ and $\hat{y}$ denotes our estimate for $y$, then we propose the following structure:

\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + H(y(t) - \hat{y}(t)) \\
\hat{y}(t) &= C\hat{x}(t) + Du(t)
\end{align*}
\]

where $H \in \mathbb{R}^{n \times m}$ is called the observer gain matrix. That these observer (estimator) equations
define a feedback structure is clear from Figure 13.2.

![Block Diagram Visualization of Constant Gain State Observer](image)

Figure 13.2: Block Diagram Visualization of Constant Gain State Observer

Given the above, if one defines the the state estimation error $\hat{x}$ and the output estimation error $\hat{y}$ as follows:

\[
\hat{x}(t) \overset{\text{def}}{=} x(t) - \hat{x}(t) \quad (13.3)
\]
\[
\hat{y}(t) \overset{\text{def}}{=} y(t) - \hat{y}(t) \quad (13.4)
\]

then the state estimation error dynamics are given as follows:

\[
\dot{\hat{x}}(t) = (A - HC)\hat{x}(t) \quad (13.5)
\]
\[
\dot{\hat{y}}(t) = C\hat{x}(t) \quad (13.6)
\]

This shows that the stability of the state estimation error dynamics is determined by the matrix $A - HC$. Specifically, since

\[
\hat{x}(t) = e^{(A-HC)t} \hat{x}(0) \quad (13.7)
\]
\[
\hat{y}(t) = Ce^{(A-HC)t} \hat{x}(0) \quad (13.8)
\]

it follows that the state estimation error $\hat{x}(\cdot)$ (and the output estimation error $\hat{y}(\cdot)$) converges to zero as $t$ approaches infinity if and only if the eigenvalues of $A - HC$ all have negative real parts.
The state space representation for the above closed loop observer structure is as follows:

\[
\dot{x} = (A - HC)\dot{x} + (B - HD)u + Hy \quad (13.9)
\]

\[
\hat{y} = C\dot{x} + Du. \quad (13.10)
\]

This shows that the eigenvalues of \( A - HC \) are precisely the poles of the closed loop observer in Figure 13.2. Given this, it is important to present a result which is analogous to the state feedback pole placement result discussed above. It can be shown that the observer gain matrix \( H \) may be selected to place the closed loop observer poles at arbitrary locations in the s-plane (with complex poles occurring in conjugate pairs) if and only if the pair \( (A, C) \) is observable [8], [26]. Moreover, it can be shown that the original system \( (A, B, C, D) \) is state reconstructible (detectable) via constant gain observer if and only if all of the unstable poles of \( A \) are observable. Many authors take this as the definition for a detectable LTI system \( (A, B, C, D) \) [26]. Unstable modes which are not observable are said to be undetectable.
Chapter 14

Fundamental System Invariance Properties

It is useful to know when important properties remain unchanged under certain operations.

For example, linear invertible transformations preserve eigenvalues (not eigenvectors), transmission zeros (not zero directions), controllability, and observability.

It can be shown that the pair \((A, B)\) is controllable if and only if \((A - BG, B)\) is controllable for some \(G\). This implies that the controllability properties of an LTI system \((A, B, C, D)\) are invariant under constant gain full state feedback. It can be shown that constant gain full state feedback does not necessarily preserve observability properties.

Analogously, it can be shown that the pair \((A, C)\) is observable if and only if \((A - HC, C)\) is observable for some \(H\). This implies that the observability properties of an LTI system \((A, B, C, D)\) are invariant under constant gain state estimation feedback. It can be shown that constant gain state estimation feedback does not necessarily preserve controllability properties.
Chapter 15

Model Based Compensators And The Separation Principle

Consider an LTI plant $P$ with state space quad $(A, B, C, D)$. Consider the feedback system in Figure 9.1 with a compensator $K$ whose structure is obtained by combining the observer/state feedback ideas discussed above. Loosely speaking, this means that an observer is used to estimate the plant state and the estimate $\hat{x}$ is used to generate a state feedback-like control law $u$. More specifically, consider a compensator $K$ described by the following equations:

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + Bu - H(e + \hat{y}) \\
\hat{y} &= C\hat{x} + Du \\
u &= -G\hat{x}.
\end{align*}
\]  

(15.1)  
(15.2)  
(15.3)

Because this compensator is based on a model of the plant $P$, it is called a model based compensator. This compensator may be visualized as shown in Figure 15.1.

The state space representation for $K$ is given by:

\[
\begin{align*}
\dot{\hat{x}} &= (A - BG - H(C - DG))\hat{x} - He \\
u &= -G\hat{x}.
\end{align*}
\]  

(15.4)  
(15.5)

From this, it follows that this compensator has a transfer function matrix given by:

\[
K(s) = G(sI - A + BG + H(C - DG))^{-1}H.
\]

(15.6)
The poles of this compensator are the eigenvalues of the matrix $A - BG - HC$. Because
\[
\begin{bmatrix}
    sI - A + BG + H(C - DG) & -H \\
    0 & G
\end{bmatrix} =
\begin{bmatrix}
    I & B \\
    0 & I
\end{bmatrix}
\begin{bmatrix}
    sI - A & -H \\
    G & 0
\end{bmatrix}
\begin{bmatrix}
    I \\
    -(C - DG) & 0
\end{bmatrix},
\]
(15.7)
it follows that the transmission zeros of $K$ are those of the state space triple $(A, H, G)$. The state space representation for the closed loop system Figure 9.1 - with the LTI plant $P = (A, B, C, D)$ and the LTI model based compensator $K = (A - BG - H(C - DG), -H, -G)$ - is given by:

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{\hat{x}}
\end{bmatrix} =
\begin{bmatrix}
    A & -BG \\
    HC & A - BG - HC
\end{bmatrix}
\begin{bmatrix}
    x \\
    \dot{x}
\end{bmatrix} +
\begin{bmatrix}
    0 & B & 0 & 0 \\
    -H & 0 & H & H
\end{bmatrix}
\begin{bmatrix}
    r \\
    d_i \\
    d_o \\
    n
\end{bmatrix},
\]
(15.8)

\[
y = [C \ 0] [x] + [0 \ 0 \ I \ 0] [r \ d_i \ d_o \ n].
\]
(15.9)

If one defines the quantity
\[
\hat{x} \overset{\text{def}}{=} x - \hat{x}
\]
(15.10)

then the state space representation for the closed loop system Figure 9.1 may be rewritten as
follows:

$$
\begin{bmatrix}
\dot{x} \\
\dot{\tilde{x}}
\end{bmatrix} =
\begin{bmatrix}
A - BG & BG \\
0 & A - HC
\end{bmatrix}
\begin{bmatrix}
x \\
\tilde{x}
\end{bmatrix} +
\begin{bmatrix}
0 & B & 0 & 0 \\
H & B & -H & -H
\end{bmatrix}
\begin{bmatrix}
r \\
d_i \\
d_o \\
n
\end{bmatrix}
$$ (15.11)

$$
y =
\begin{bmatrix}
C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\tilde{x}
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & I & 0
\end{bmatrix}
\begin{bmatrix}
r \\
d_i \\
d_o \\
n
\end{bmatrix}.
$$ (15.12)

From the above, it follows that the closed loop poles of the feedback system under consideration are precisely the eigenvalues of $A - BG$ and those of $A - HC$. This observation is called the separation principle [8]. It implies that as far as closed loop pole placement is concerned, we can design the control gain matrix $G$ independently from the observer gain matrix $H$. Methods for computing $G$ and $H$ are discussed in [8], [26], [28], [29], [50], [51].
Chapter 16

Representing Uncertainty

Uncertainty is present in all dynamical systems. Parameters are uncertain. This is called parametric uncertainty. Dynamics are also uncertain; i.e. there always exists extra dynamics (modes) which have not been accounted for. This is called dynamical uncertainty.

Because uncertainty must be accounted for in system analysis and design, and specifically, in the design of feedback controllers, it is useful to develop models for representing such uncertainty. Several commonly used models are now presented. In the discussion which follows \( P_0 \) represents a nominal model for a plant \( P \) and \( \Delta \) represents uncertainty.

Additive Uncertainty. Perhaps the most obvious way to represent uncertainty is via the following additive uncertainty model:

\[
P = P_0 + \Delta. \quad (16.1)
\]

This model may be visualized as shown in Figure 16.1.
Figure 16.1: Additive Uncertainty

**Multiplicative Uncertainty At Plant Input.** Uncertainty may be modeled at the plant input using the following **multiplicative uncertainty model**:

\[ P = P_o[I + \Delta]. \] \hspace{1cm} (16.2)

This model may be visualized as shown in Figure 16.2.

Figure 16.2: Multiplicative Uncertainty At Plant Input

**Multiplicative Uncertainty At Plant Output.** Uncertainty may be modeled at the plant output using the following **multiplicative uncertainty model**:

\[ P = [I + \Delta]P_o. \] \hspace{1cm} (16.3)

This model may be visualized as shown in Figure 16.3.
**Divisive Uncertainty At Plant Input.** Uncertainty may be modeled at the plant input using the following *divisive uncertainty model*:

\[ P = P_0 [I + \Delta]^{-1}. \]  

(16.4)

This model may be visualized as shown in Figure 16.4.

**Figure 16.4: Divisive Uncertainty At Plant Input**

**Divisive Uncertainty At Plant Output.** Uncertainty may be modeled at the plant output using the following *divisive uncertainty model*:

\[ P = [I + \Delta]^{-1} P_0. \]  

(16.5)

This model may be visualized as shown in Figure 16.5.
Further discussion on uncertainty may be found in [50], [51].
Chapter 17

A New Paradigm For Control System Design

It is often desirable to design feedback controllers which exploit many measurements. Toward this end, consider the general feedback system in Figure 17.1.

In this figure

- $P$ represents a *generalized plant* which may include a nominal plant model, actuator dynamics, spectral shaping filters, sensor dynamics, and weighting functions;

- $K$ represents a *compensator* or *controller* to be designed on the basis of a nominal model $P_0$ for the generalized plant $P$ and some knowledge of the uncertainty in modeling $P$ by $P_0$;
• $w$ represents *exogenous signals* (e.g. reference commands, disturbances, sensor noise, etc.);

• $z$ represents signals to be *regulated* or kept small (e.g. tracking errors, control signals, etc.);

• $u$ represents *control signals* (e.g. gas flow, elevator deflection, etc.) which are to be generated by the compensator $K$ so as to ensure that $r$ is “sufficiently small.”

*Generalized Plant Model.* If the generalized plant $P$ represents an LTI system, then it may be appropriate to model it as follows:

$$
\dot{x} = Ax + B_1w + B_2u \quad (17.1)
$$

$$
z = C_{11}x + D_{11}w + D_{12}u \quad (17.2)
$$

$$
y = C_{21}x + D_{21}w + D_{22}u \quad (17.3)
$$

For notational economy, the above state space representation often written as follows:

$$
P = \begin{bmatrix}
A & B_1 & B_2 \\
C_{11} & D_{11} & D_{12} \\
C_{21} & D_{21} & D_{22}
\end{bmatrix} \quad (17.4)
$$

*Compensator Model.* Similarly, if the compensator $K$ represents an LTI system, then it may be appropriate to model it as follows:

$$
\dot{x}_k = A_kx_k + B_ky \quad (17.5)
$$

$$
u = C_kx_k + D_ky. \quad (17.6)
$$

Once again, for notational economy this is often written as follows:

$$
K = \begin{bmatrix}
A_k & B_k \\
C_k & D_k
\end{bmatrix} \quad (17.7)
$$

*Closed Loop System.* It can also be shown that

$$
u = [I - D_kD_{22}]^{-1}C_{21}x + [I - D_kD_{22}]^{-1}C_kx_k + [I - D_kD_{22}]^{-1}D_kD_{21}w. \quad (17.8)
$$

From this, one obtains the state space representation for the closed loop system as follows:

$$
T_{wz} = \begin{bmatrix}
A + B_2[1 - D_kD_{22}]^{-1}D_kC_{21} & B_2[1 - D_kD_{22}]^{-1}C_k & B_2[1 - D_kD_{22}]^{-1}D_kD_{21} \\
C_{11} + D_{12}[1 - D_kD_{22}]^{-1}D_kC_{21} & A + D_{11}[1 - D_kD_{22}]^{-1}C_k & D_{11}[1 - D_kD_{22}]^{-1}D_kD_{21} \\
C_{21} + D_{22}[1 - D_kD_{22}]^{-1}D_kC_{21} & D_{21}[1 - D_kD_{22}]^{-1}C_k & D_{21}[1 - D_kD_{22}]^{-1}D_kD_{21}
\end{bmatrix} \quad (17.9)
$$
It can also be shown that

\[
y = [C_{21} + D_{22}[I - D_kD_{22}]^{-1}D_kC_{21}]x + [D_{22}[I - D_kD_{22}]^{-1}C_kx_k + [D_{21} + D_{22}[I - D_kD_{22}]^{-1}D_kD_2]u7.10
\]

Control system analysis and design techniques which focus on the above structure are addressed in [50], [51].
Chapter 18

State Space Models For Time Varying Systems

The focus thus far has been on LTI systems and specifically on systems described by linear ordinary differential equations with constant coefficients. In practice, however, it may be that the coefficients vary with time (e.g. mass of a rocket). This motivates the study of continuous time systems possessing the following state space representation:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t); \quad x(0^-) = x_0 \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]

where \(A(\cdot) \in \mathbb{R}^{n \times n}, B(\cdot) \in \mathbb{R}^{n \times m}, C(\cdot) \in \mathbb{R}^{p \times n}, D(\cdot) \in \mathbb{R}^{p \times m}\), denote matrix functions of the independent continuous variable \(t \in \mathbb{R}\), \(u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m\) denotes a vector of \(m\) input variables, \(x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n\) denotes a vector of \(n\) state variables, \(y = [y_1, \ldots, y_p]^T \in \mathbb{R}^p\) denotes a vector of \(p\) output variables, and \(x_0 = [x_{1a}, \ldots, x_{na}]^T \in \mathbb{R}^n\) denotes a vector of \(n\) initial conditions for the state. Such systems are called linear time varying (LTV) systems. Many of the concepts presented thus far extend to LTV systems [2], [6], [8], [26], [28], [40], [45], [46].
Chapter 19

Nonlinear Systems, Equilibria, And Linearization

In practice, linear dynamical models are often used as approximations to nonlinear dynamical models - in the same fashion that lines are used to approximate nonlinear functions in calculus. To see how such linear approximations are obtained, we begin by considering nonlinear systems possessing the following nonlinear state space representation:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) & x(0) &= x_0 \\
y(t) &= g(x(t), u(t))
\end{align*}
\]

(19.1)

(19.2)

where \( f = [f_1(x_1, \ldots, x_n, u_1, \ldots, u_m), \ldots, f_n(x_1, \ldots, x_n, u_1, \ldots, u_m)]^T \in \mathcal{R}^n \) denotes a vector of \( n \) functions, \( g = [g_1(x_1, \ldots, x_n, u_1, \ldots, u_m), \ldots, g_p(x_1, \ldots, x_n, u_1, \ldots, u_m)]^T \in \mathcal{R}^p \) denotes a vector of \( p \) functions, \( u = [u_1, \ldots, u_m]^T \in \mathcal{R}^m \) denotes a vector of \( m \) input variables, \( x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n \) denotes a vector of \( n \) state variables, \( y = [y_1, \ldots, y_p]^T \in \mathcal{R}^p \) denotes a vector of \( p \) output variables, and \( x_o = [x_{o_1}, \ldots, x_{o_n}]^T \in \mathcal{R}^n \) denotes a vector of \( n \) initial conditions for the state.

**Equilibrium.** A point \((x_e, u_e)\) is said to be an *equilibrium* [27], [47] of the above nonlinear system at \( t = 0 \) if

\[
f(x, u) = 0 \quad \text{for all } t \geq 0.
\]

(19.3)

**Linearization.** A linear state space representation which approximates the above system near
\((x_e, u_e)\) is obtained as follows:

\[
\delta x(t) = A\delta x(t) + B\delta u(t) \quad \delta x(0) = \delta x_o \quad (19.4)
\]

\[
\delta y(t) = C\delta x(t) + D\delta u(t) \quad (19.5)
\]

where

\[
A = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}_{(x_e, u_e)} 
B = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m}
\end{bmatrix}_{(x_e, u_e)} \quad (19.6)
\]

\[
C = \begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n}
\end{bmatrix}_{(x_e, u_e)} 
D = \begin{bmatrix}
\frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_n}{\partial u_1} & \cdots & \frac{\partial g_n}{\partial u_m}
\end{bmatrix}_{(x_e, u_e)} \quad (19.7)
\]

\[
\delta u(t) \overset{\text{def}}{=} u(t) - u_e \quad (19.8)
\]

\[
\delta x(t) \overset{\text{def}}{=} x(t) - x_e \quad (19.9)
\]

\[
\delta y(t) \overset{\text{def}}{=} y(t) - y_e \quad (19.10)
\]

\[
\delta x_o \overset{\text{def}}{=} x_o - x_e \quad (19.11)
\]

\[
y_e \overset{\text{def}}{=} g(x_e, u_e). \quad (19.12)
\]

The above linear dynamical model is an accurate representation for the original nonlinear dynamical system provided that the initial state perturbation \(\delta x_o\) and the control perturbation \(\delta u(\cdot)\) are sufficiently small.
Chapter 20
An Introduction To Discrete Time Signals

With the advent of affordable desktop computing, digital computers with state-of-the-art analysis and design software [11], [21], [30] is becoming widely accessible. Because all digital data formats are here to stay, it is very desirable to develop mathematical tools that are directly applicable. Toward this end, we seek tools that are appropriate for dealing with the discrete nature of computer data.

20.1 Sampling And Discrete Time Signals

We begin by considering the continuous time signal \( f(\cdot) \). Such a signal, in general, consists of infinitely many points. Because memory is always limited, one cannot possibly store all of the data points which make up \( f \). Instead, it is realistic to store “samples” of the original signal. This motivates the concepts of sampling and sampled signals. If \( f \) is sampled every \( T_s > 0 \) time units, one obtains the continuous time sampled signal \( f_s \):

\[
f_s(t) \overset{\text{def}}{=} \begin{cases} 
  f(t) & t = nT_s \\
  0 & \text{elsewhere}
\end{cases} \quad n = 0, \pm 1, \pm 2, \ldots \quad (20.1)
\]

The quantity \( T_s \) is called the sampling period or the sampling time. The sampling frequency is defined to be

\[
f_s \overset{\text{def}}{=} \frac{1}{T_s} \quad (20.2)
\]
with units (1/unit time) or
\[
\Omega_s \overset{\text{def}}{=} \frac{2\pi}{T_s}
\] (20.3)

with units (radians/unit time). Since the signal \(f_s\) takes on nonzero values only at integer multiples of \(T_s\), it makes sense to redefine the independent variable. For notationally economy we thus define the discrete time signal
\[
f[n] \overset{\text{def}}{=} \begin{cases} 
  f(nT_s) & n = 0, \pm 1, \pm 2, \ldots \\
  \text{undefined} & \text{otherwise}
\end{cases}
\] (20.4)

Because \(f[\cdot]\) is defined only for integer values of \(n\), we often just write:
\[
f[n] \overset{\text{def}}{=} f(nT_s) \quad n = 0, \pm 1, \pm 2, \ldots
\] (20.5)

It must be emphasized that such signals are discrete in time but continuous in amplitude. That is, they can assume any real value and each \(n\). This is in contrast with real-world digital signals which are discrete in time and discrete in amplitude [35].

### 20.2 Kronecker Delta Function

Just as the Dirac delta distribution is of fundamental significance in the development of continuous time signal and systems theory, in discrete time we have the equally significant Kronecker delta function defined as follows:
\[
\delta[n] \overset{\text{def}}{=} \begin{cases} 
  1 & n = 0 \\
  0 & \text{elsewhere}
\end{cases}
\] (20.6)

Conceptually, this discrete time delta function is far simpler than the Dirac distribution which we recall was defined primarily by an integral sifting property.
20.3 Unit Step Function

Analogous to continuous time, the discrete time unit step function is defined as follows:

\[ 1[n] \overset{\text{def}}{=} \begin{cases} 
1 & n = 0, 1, \ldots \\
0 & \text{elsewhere}
\end{cases} \]  
(20.7)

20.4 Real Exponential Signals

In discrete time applications, it is useful to consider real exponential test signals such as

\[ f[n] = a^n 1[n] \]  
(20.8)

where \( a \geq 0 \). This function represents a growing exponential for \( a > 1 \), a unit step for \( a = 1 \), and a decaying exponential for \( a \in [0, 1) \).

20.5 Complex Exponential Signals: Creating Discrete Time Sinusoids and Exponential Sinusoids

As in continuous time, discrete time sinusoids and exponential sinusoids may be constructed from complex exponentials as follows:

\[ |X| r_o^n \cos(\omega_o n + \angle X) \ 1[n] = \text{Re}\{ |X| r_o^n e^{j(\omega_o n + \angle X)} \} 1[n] = \text{Re}\{ |X| e^{j\angle X} [r_o e^{j\omega_o}]^n \} 1[n] \]  
(20.9)

\[ |X| r_o^n \sin(\omega_o n + \angle X) \ 1[n] = \text{Im}\{ |X| r_o^n e^{j(\omega_o n + \angle X)} \} 1[n] = \text{Im}\{ |X| e^{j\angle X} [r_o e^{j\omega_o}]^n \} 1[n] \]  
(20.10)

These signals possess a complex frequency \( z_o = r_o e^{j\omega_o} \) and a conjugate frequency \( \bar{z}_o = r_o e^{-j\omega_o} \). These signals decay to zero as \( n \) approaches \( \infty \) if and only if \( |z_o| = r_o \in [0, 1) \). They maintain a constant amplitude for all \( n \) if and only if \( |z_o| = r_o = 1 \). They grow in magnitude as \( n \) approaches \( \infty \) if and only if \( |z_o| = r_o \in (1, \infty) \). The above simple analysis immediately reveals the following correspondence relationships between discrete time and continuous time:

- unit circle in \( z \)-plane (\(|z| = 1\)) corresponds to imaginary axis in \( s \)-plane (\( s = j\Omega \)),

- interior of unit circle in \( z \)-plane (\(|z| < 1\)) corresponds to left half \( s \)-plane (\( Re \ s < 0 \)),

- exterior of unit circle in \( z \)-plane (\(|z| > 1\)) corresponds to right half \( s \)-plane (\( Re \ s > 0 \)),
• exterior of unit circle in $z$-plane ($|z| > 1$) corresponds to right half $s$-plane ($Im \ s > 0$).

It will be seen that the above correspondence relationships provide a natural connection between the continuous time concepts presented earlier and the discrete time concepts to be presented.

*Frequency Periodicity of Discrete Time Sinusoids.* Unlike sinusoids in continuous time, discrete time sinusoids are periodic in $\omega_0$ with period $2\pi$. This implies that discrete time sinusoids whose frequencies differ by an integer multiple of $2\pi$ are identical to one another. This is in sharp contrast with continuous time sinusoids.
Chapter 21

An Introduction To Discrete Time Systems: Discretization Of Continuous Time LTI Systems

Consider the continuous time LTI system described by the following state space representation:

\[
\begin{align*}
\dot{x} &= A_c x + B_c u \\
y &= C_c x + D_c u.
\end{align*}
\]  
\tag{21.1}

While equations (7.31) and (7.32) provide expressions for \(x\) and \(y\) in terms of \(x_0\) and \(u\), it is still useful to consider how the above continuous time state space representation might be solved numerically within a digital computer.

One natural approach to consider is to approximate the derivative \(\dot{x}\) by a finite difference [22] as follows:

\[
\dot{x}(nT_s) \approx \frac{x((n + 1)T_s) - x(nT_s)}{T_s}
\]  
\tag{21.3}

where \(T_s > 0\) is the sampling time and is chosen sufficiently small such that the approximation is “good enough”. Substituting this approximation into the state equation yields:

\[
\dot{x}(nT_s) \approx \frac{x((n + 1)T_s) - x(nT_s)}{T_s} = A_c x(nT_s) + B_c u(nT_s).
\]  
\tag{21.4}

From this, it then follows that

\[
x((n + 1)T_s) = (1 + T_s A_c)x(nT_s) + T_s B_c u(nT_s)
\]  
\tag{21.5}
\begin{equation}
y(nT_s) = C_c x(nT_s) + D_c u(nT_s).
\end{equation}

This then yields the following:

\begin{equation}
x[n + 1] = (1 + T_s A_c)x[n] + T_s B_c u[n] \quad x[0] = x_0
\end{equation}

\begin{equation}
y[n] = C_c x[n] + D_c u[n].
\end{equation}

These equations may be rewritten to obtain the following \textit{discrete time state space representation}:

\begin{equation}
x[n + 1] = A x[n] + B u[n] \quad x[0] = x_0
\end{equation}

\begin{equation}
y[n] = C x[n] + D u[n].
\end{equation}

We say that the continuous time system has been \textit{discretized} to obtain the above discrete time system. The above provides just one motivate for the study of discrete time systems.

Equation (21.9) defines a system of \textit{discrete time ordinary difference equations with constant coefficients}. Proceeding recursively, one obtains the following relationships for \(x[\cdot]\) and \(y[\cdot]\):

\begin{equation}
x[n] = \begin{cases}
x_0 & n = 0 \\
A x_0 + B u[0] & n = 1 \\
A^n x_0 + \sum_{k=0}^{n-1} A^{n-1-k} B u[k] & n = 2, 3, \ldots
\end{cases}
\end{equation}

\begin{equation}
y[n] = \begin{cases}
C x_0 + D u[0] & n = 0 \\
C A x_0 + C B u[0] + D u[1] & n = 1 \\
C A^n x_0 + \sum_{k=0}^{n-1} C A^{n-1-k} B u[k] + D u[n] & n = 2, 3, \ldots
\end{cases}
\end{equation}

For notational economy, it is often convenient to write the above expressions more concisely as follows:

\begin{equation}
x[n] = A^n x_0 + \sum_{k=0}^{n-1} A^{n-1-k} B u[k]
\end{equation}

\begin{equation}
y[n] = C A^n x_0 + \sum_{k=0}^{n-1} C A^{n-1-k} B u[k] + D u[n].
\end{equation}

As with continuous time signals and systems, a proper treatment of discrete time signals and systems requires suitable transform techniques.
Chapter 22

An Introduction To $\mathcal{Z}$-Transforms

In what follows, we will study systems described by linear ordinary difference equations with constant coefficients. Given this, an indispensable tool is the unilateral $\mathcal{Z}$-transform [33], [34]. A precise definition is now provided.

Definition: Unilateral $\mathcal{Z}$-Transform.

The unilateral $\mathcal{Z}$-transform of a discrete time function $f[\cdot]$ is denoted $F(z) = (\mathcal{Z}f)[n]$ and defined as follows:

$$F(z) = (\mathcal{Z}f)[n] \overset{\text{def}}{=} \sum_{n=0}^{\infty} f[n]z^{-n} \quad (22.1)$$

The set of $z$ values for which the $\mathcal{Z}$-transform summation converges (i.e. makes sense) is called the region of convergence (ROC) of $F$ [33]. A list of elementary $\mathcal{Z}$-transform pairs is contained in Table 22.1. A comprehensive list of $\mathcal{Z}$-transform properties is contained in Table 22.2.

Geometric Series. In the study of discrete time signals and systems, geometric series are regularly encountered. Given this, consider the finite geometric series

$$S_N = \sum_{k=0}^{N-1} ar^k \quad (22.2)$$

where

- $N$ is the number of terms in the series,
\[
\begin{array}{|c|c|c|}
\hline
f[n] & F(z) & \text{ROC} \\
\hline
1. & \delta[n] & 1 & \text{All } z \\
2. & a^n 1[n] & \frac{1}{1 - az^{-1}} & |z| > |a| \\
3. & r^n \cos(\omega_c n + \theta) 1[n] & z \left[ \frac{\cos(\theta) - rz \cos(\omega_c + \theta)}{1 - 2rz \cos(\omega_c + \theta) + r^2} \right] & |z| > |r_c| \\
4. & r^n \sin(\omega_c n + \theta) 1[n] & z \left[ \frac{\sin(\theta) - rz \sin(\omega_c + \theta)}{1 - 2rz \cos(\omega_c + \theta) + r^2} \right] & |z| > |r_c| \\
\hline
\end{array}
\]

Table 22.1: Elementary \(Z\)-Transform Pairs

\[
\begin{array}{|c|c|c|}
\hline
\text{n-Description} & \mathbf{f}[n] & \mathbf{F(z)} & \mathbf{z-Description} \\
\hline
\text{Linearity} & a_1 f_1[n] + a_2 f_2[n] & a_1 F_1(z) + a_2 F_2(z) & \text{Linearity} \\
\text{Exponentiation} & a^n f[n] & F\left(\frac{z}{a}\right) & \text{Scaling} \\
\text{Multiplication} & n^n f[n] & (-z\frac{dz}{dz})^n (F(z)) & \text{Differentiation} \\
\text{Shift (Delay)} & f[n - m] 1[n - m] & z^{-m} F(s); \ m = 0, 1, \ldots & \text{Division} \\
\text{Leftward Shift} & f[n + m] 1[n] & z^m \left[ F(z) - \sum_{k=0}^{m-1} f[k]z^{-k} \right] & \text{Multiplication} \\
\text{Rightward Shift} & f[n - m] 1[n] & z^{-m} \left[ F(z) + \sum_{k=m}^{\infty} f[k]z^{-k} \right] & \text{Division} \\
\text{Summation} & \sum_{k=0}^{n} f[k] & \frac{1}{1 - z^{-1}} F(z) & \text{Function Multiplication} \\
\text{Convolution} & (f_1 * f_2)[n] \overset{\text{def}}{=} \sum_{k=0}^{n} f_1[n-k]f_2[k] & F_1(z)F_2(z) & \text{Function Multiplication} \\
\hline
\end{array}
\]

Table 22.2: \(Z\)-Transform Properties

- \(a\) is the first term in the series, and
- \(r\) is the common ratio.

Noting that
\[
S_N - rS_N = \sum_{k=0}^{N-1} ar^k - \sum_{k=0}^{N-1} ar^{k+1} = a \left[ 1 - ar^N \right],
\]

it follows that
\[
S_N = \sum_{k=0}^{N-1} ar^{k-1} = a \left[ \frac{1 - ar^N}{1 - r} \right]. \tag{22.4}
\]

For \(N = \infty\), it follows that
\[
S_\infty = \sum_{k=0}^{\infty} ar^{k-1} = \left[ \frac{a}{1 - r} \right] \text{ if and only if } |r| < 1. \tag{22.5}
\]
With $\mathcal{Z}$-transforms in place, one can now extend all of the continuous time concepts to discrete time.
Chapter 23

Linear Shift Invariant Systems With State Space Representations

Consider a discrete time system with state space representation

\[
x[n + 1] = Ax[n] + Bu[n] \quad x[0] = x_0
\]
\[
y[n] = Cx[n] + Du[n]
\]

with \( u = [u_1, \ldots, u_m]^T \in \mathcal{R}^m \) denotes a vector of \( m \) input variables, \( x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n \) denotes a vector of \( n \) state variables, \( y = [y_1, \ldots, y_p]^T \in \mathcal{R}^p \) denotes a vector of \( p \) output variables, and \( x_0 = [x_{1_0}, \ldots, x_{n_0}]^T \in \mathcal{R}^n \) denotes a vector of \( n \) initial conditions for the state. Taking Z-transforms yields the following for \( X \) and \( Y \):

\[
X(z) = z(zI - A)^{-1}x_0 + (zI - A)^{-1}BU(z)
\]
\[
Y(z) = Cz(zI - A)^{-1}x_0 + [C(zI - A)^{-1}B + D]U(z).
\]

23.1 Transfer Function Matrix

Given this, the transfer function matrix from the input vector \( u(\cdot) \) to the output vector \( y(\cdot) \) is given by

\[
H(z) = C(zI - A)^{-1}B + D.
\]
23.2 Poles and Characteristic Equation

The system poles are the roots of the characteristic equation

$$\Phi(z) = \det(zI - A).$$

(23.8)

(23.9)

23.3 Exponential Stability

The discrete time system \((A, B, C, D)\) is exponentially stable if and only if all of its poles lie within the unit circle \((|z| < 1)\).

23.4 Delta Response Matrix, Convolution

The delta response matrix \(h[\cdot]\) is the response to a unit \(\delta\) signal in each channel. It is also the inverse \(Z\)-transform of the transfer function matrix \(H(\cdot)\) and is given by

$$h[n] = CA^{n-1}B1[n - 1] + D\delta[n].$$

(23.10)

From equation (21.14), it follows that

$$y[n] = CA^n x_o + \sum_{k=0}^{n-1} CA^{n-1-k} Bu[k] + Du[n]$$

(23.11)

$$= CA^n x_o + \sum_{k=0}^{n-1} h[n - k] u[k]$$

(23.12)

$$= CA^n x_o + (h * u)[n]$$

(23.13)

where \((h * u)[\cdot]\) denotes the discrete time convolution of \(h[\cdot]\) with \(u[\cdot]\). The first term in the above expression is due to the initial condition vector \(x_o\) and is called the zero input response (ZIR). The second term (convolution term) is due to the input and is called the zero state response (ZSR).
23.5 Frequency Response Matrix: Periodicity

The *frequency response matrix* of the system is specified by the transfer function matrix evaluated on the unit circle ($|z| = 1$); i.e.

$$H(e^{j\omega}) = C(e^{j\omega}I - A)^{-1} B + D \quad (23.14)$$

where $\omega > 0$. This follows from the fact that the unit circle in the discrete time setting corresponds to the imaginary axis in the continuous time setting. The matrix $H(e^{j\omega})$ is also the *Discrete Time Fourier Transform* of $h[\cdot]$ [34], [35], [36]; i.e.

$$H(e^{j\omega}) \overset{def}{=} \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}. \quad (23.15)$$

It is sometimes called the *frequency spectrum* or *spectrum of* $h[\cdot]$. It should be noted that discrete time frequency responses and spectrums are periodic in $\omega$ with period $2\pi$. This is in sharp contrast to continuous time frequency responses and spectrums which, in general, are not periodic.

For SISO systems, $H(\cdot)$ is a scalar transfer function. For such systems, the *magnitude response* is given by $|H(e^{j\omega})|$ and the *angle response* is given by $\angle H(e^{j\omega})$.

23.6 Input-Output and State Space Representations, Ordinary Difference Equation

For SISO discrete time systems with a transfer function $H(\cdot)$, the system’s associated input-output linear ordinary difference equation may be readily obtained from the system transfer function. This is done by exploiting the (zero initial condition) transform pair:

$$f[n + m] \longleftrightarrow z^m F(z). \quad (23.16)$$

For MIMO discrete time systems $(A, B, C, D)$, techniques similar to those used for continuous time systems may be used to obtain an input-output difference equation.
Similarly, to obtain a state space representation from a transfer functions, the continuous time techniques are directly applicable with $s$ replaced by $z$.

**Linearity.** The MIMO discrete time system $(A, B, C, D)$ is said to be linear because $x$ and $y$ are linear in the vector $[ x_o^T \ u^T (\cdot)]^T$.

### 23.7 Shift Invariance

The MIMO discrete time system $(A, B, C, D)$ is said to be *shift invariant* because (under zero initial conditions) it commutes with the *unilateral shift operator* $z^{-1}$ defined by the relationship:

\[
(S_z w)[n] \overset{\text{def}}{=} w[n - 1]
\]  

(23.17)

The unilateral shift thus represents a single sample delay operator.

### 23.8 Linear Shift Invariant (LSI) Systems

Because they are linear and shift invariant, the discrete time systems $(A, B, C, D)$ are classified as *linear shift invariant (LSI) systems*.

### 23.9 Sinusoidal Steady State Analysis

Consider an exponentially stable SISO discrete time system with transfer function $H(\cdot)$ corresponding to a real impulse response $h[\cdot]$. Suppose that an input $u[n] = |X|r_o^n \cos(\omega_o n + \angle X)$ is applied to the system. The corresponding steady state output is given by $y_{ss} = |X|r_o^n |H(r_o e^{j\omega_o})| \cos(\omega_o n + \angle X + \angle H(r_o e^{j\omega_o}));$ i.e.

\[
u[n] = |X|r_o^n \cos(\omega_o n + \angle X) \quad \overset{\text{produces}}{\longrightarrow} \quad y_{ss} = r_o^n |H(r_o e^{j\omega_o})| |X| \cos(\omega_o n + \angle X + \angle H(r_o e^{j\omega_o})|)
\]  

(23.18)
From this, it follows that

\[ u[n] = |X| \cos(\omega_0 n + \angle X) \quad \text{produces} \quad y_{ss} = |H(e^{j\omega_0})||X| \cos(\omega_0 n + \angle X + \angle H(e^{j\omega_0})). \] (23.19)

This relationship provides the key to steady state sinusoidal analysis for discrete time systems.

The remainder of the continuous time concepts extend readily to discrete time.
Chapter 24


It is desirable to understand the sampling process from a frequency domain perspective. Toward this end, we now propose the following model for the sampling process.

24.1 Impulse Sampling Model For Sampling Process

Suppose that \( f_C(\cdot) \) represents a continuous time signal. To form a model for the sampling process, consider the infinite train of continuous time impulses (Dirac deltas)

\[
\sum_{n=-\infty}^{\infty} \delta(t - nT_s)
\]

where \( T_s > 0 \) is the sampling period. No consider the product of the continuous time signal \( f_C(\cdot) \) with this train to obtain the impulse sampled signal:

\[
f_{c,s}(t) \overset{\text{def}}{=} f_C(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} f_C(nT_s) \delta(t - nT_s).
\]

This signal represents a train of “weighted” (impulse modulated) Dirac delta functions. As such, \( f_{c,s}(\cdot) \) is not equal to the sampled signal

\[
f_{c,s}(t) \overset{\text{def}}{=} \begin{cases} 
  f_C(t) & t = nT_s, \\
  0 & \text{elsewhere},
\end{cases} 
\]

where \( n = 0, \pm 1, \pm 2, \ldots \). Letting \( C/D \) denote a block which converts continuous time Dirac deltas to
discrete time Kronecker deltas, the discrete time signal obtained from the continuous time
signal \( f_c(\cdot) \) is given by
\[
f[n] \overset{\text{def}}{=} f_{c_{E_0}}(nT_s) \quad n = 0, \pm 1, \pm 2, \ldots.
\] (24.4)
Moreover, the sampling process may be visualized as indicated in Figure 24.1.

![Figure 24.1: Impulse Sampling Model For Sampling Process](image)

Let \( F_c(\cdot) \) and \( F_{c_{E_0}}(\cdot) \) denote the frequency spectrums of \( f_c(\cdot) \) and \( f_{c_{E_0}}(\cdot) \), respectively. Given this, the Discrete Time Fourier Transform or frequency spectrum of \( f[\cdot] \) is related to that of \( f_{c_{E_0}}(\cdot) \) and that of \( f_c(\cdot) \) as follows:
\[
F(e^{j\omega}) = F_{c_{E_0}}(j\frac{\omega}{T_s}) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} F_c(j\left(\frac{\omega}{T_s} - n\Omega_s\right)) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} F_c(j\left(\frac{\omega - n2\pi}{T_s}\right))
\] (24.5)
where \( \Omega_s \overset{\text{def}}{=} \frac{2\pi}{T_s} \) is the sampling frequency. This equation shows that sampling results in spectral shifting and duplication. More precisely, it shows that the spectrum of \( f[\cdot] \) is obtained by scaling that of \( f_{c_{E_0}}(\cdot) \) which in turn is obtained by scaling the spectrum of \( f_c(\cdot) \) by \( T_s \), shifting it integer multiples of the sampling frequency \( \Omega_s \), and summing the duplicates.

**Aliasing, Signal Reconstruction, and the Whittaker-Shannon-Nyquist Sampling Theorem.** When the duplicates overlap, we say that spectral aliasing or aliasing has occurred. If the spectrum of \( f_c(\cdot) \) is limited to frequencies below \( \Omega_{max} \) and if the sampling frequency \( \Omega_s \) is chosen such that
\[
\Omega_s > \Omega_{NR} \overset{\text{def}}{=} 2\Omega_{max},
\] (24.6)
then spectral aliasing will not occur and the original continuous time signal \( f_c(\cdot) \) can be reconstructed from its discrete time samples \( f[\cdot] \) using an ideal low pass filter with cut-off frequency \( \Omega_c = \Omega_{\text{max}} \) [34, pg. 514-520], [36]. More precisely,

\[
f_c(t) = \sum_{n=-\infty}^{\infty} f[n] \frac{\sin \Omega_{\text{max}} (t - nT_s)}{\Omega_{\text{max}} (t - nT_s)}.
\]  

(24.7)

This very important result is known as the Whittaker-Shannon-Nyquist Sampling Theorem [31], [42], [48]. The frequency \( \Omega_{\text{max}} \) is often called the Nyquist frequency and \( \Omega_{\text{NR}} \) the Nyquist rate.
Chapter 25

Discrete Time LSI Simulation Of Continuous Time LTI Systems

There are many ways to simulate a continuous time LTI system

\[ \dot{x}(t) = A_c x(t) + B_c u(t) \quad x(0) = x_0 \quad (25.1) \]

\[ y(t) = C_c x(t) + D_c u(t) \quad (25.2) \]

\[ H_c(s) = C_c (s I - A_c)^{-1} B_c + D_c \quad (25.3) \]

\[ h_c(t) = C_c e^{A_d t} B_c \, 1(t) + D_c \, \delta(t) \quad (25.4) \]

on a computer. Suppose that it is desirable to simulate the above system using a discrete time LSI system represented as follows:

\[ x[n+1] = A_d x[n] + B_d u[n] \quad x[0] = x_0 \quad (25.5) \]

\[ y[n] = C_d x[n] + D_d u[n] \quad (25.6) \]

\[ H_d(z) = C_d (z I - A_d)^{-1} B_d + D_d \quad (25.7) \]

\[ h_d[n] = C_d A_d^{n-1} B_d \, 1[n] + D_d \, \delta[n] \quad (25.8) \]

Let \( T_s > 0 \) denote the sampling period. Methods for selecting \([A_d, B_d, C_d, D_d] \) are now presented.
25.1 Delta Invariance Method

If one selects the delta response of the discrete time system such that

\[ h_d[n] = h_c(nT_s), \]  

(25.9)

then one obtains

\[ H_d(z) = H_c(s) \bigg|_{s = \frac{1}{T_s} \frac{A_c}{1}} \]  

(25.10)

\[ A_d = e^{A_cT_s} \]  

(25.11)

\[ B_d = B_c \]  

(25.12)

\[ C_d = C_c e^{A_cT_s} \]  

(25.13)

\[ D_d = D_c + C_cB_c. \]  

(25.14)

With this implementation, the left half s-plane gets mapped into the unit circle and exponentially stable matrices \( A_c \) get mapped into exponentially stable matrices \( A_d \).

25.2 Forward Euler Integration-Backward Difference Method

If one uses the Forward Euler Integral Approximation:

\[ x(nT_s) - x((n-1)T_s) = \int_{(n-1)T_s}^{nT_s} \dot{x}(\tau) d\tau \approx T_s \dot{x}(nT_s) \]  

(25.15)

or equivalently, the Backward Difference Approximation:

\[ \dot{x}(nT_s) \approx \frac{x(nT_s) - x((n-1)T_s)}{T_s} \]  

(25.16)

then one obtains

\[ H_d(z) = H_c(s) \bigg|_{s = \frac{1 - \frac{1}{T_s}}{T_s}} \]  

(25.17)

\[ A_d = [I - A_cT_s]^{-1} \]  

(25.18)

\[ B_d = [I - A_cT_s]^{-1} B_c \]  

(25.19)
\[ C_d = C_c T_s [I - A_c T_s]^{-1} \]
\[ D_d = D_c + C_c T_s [I - A_c T_s]^{-1} B_c. \]  

With this implementation, the left half s-plane gets mapped into a circle with center at \( z = 1/2 \) and radius \( 1/2 \). Hence, exponentially stable matrices \( A_c \) get mapped into exponentially stable matrices \( A_d \).

### 25.3 Backward Euler Integration-Forward Difference Method

If one uses the Backward Euler Integral Approximation:

\[ x(n T_s) - x((n-1) T_s) = \int_{(n-1) T_s}^{n T_s} \dot{x}(\tau) d\tau \approx T_s \dot{x}(n - 1) T_s) \]  

or equivalently, the Forward Difference Approximation:

\[ \dot{x}(n T_s) \approx \frac{x((n+1) T_s) - x(n T_s)}{T_s} \]

then one obtains

\[ H_d(z) = H_c(s) \big|_{s = -\frac{z}{T_s}} \]
\[ A_d = I + A_c T_s \]
\[ B_d = T_s B_c \]
\[ C_d = C_c \]
\[ D_d = D_c. \]

With this implementation, left half s-plane poles \( s = -\sigma + j \Omega \) get mapped to z-plane poles \( z = (1 - \sigma T_s) + j \omega T_s \). Hence, exponentially stable matrices \( A_c \) get mapped into exponentially stable matrices \( A_d \) provided that \( T_s \) is sufficiently small. If the continuous time system poles are denoted \( s_i = -\sigma_i + j \Omega_i \), then the discrete time system will be exponentially stable if and only if

\[ T_s < T_c \overset{\text{def}}{=} \min_{i = 1, \ldots, n} \frac{2 \sigma_i}{\sigma_i^2 + \Omega_i^2}. \]
In practice, one usually selects $T_s$ to be much smaller than $T_c$. Choosing $T_s = 0.1T_c$ is a good rule of thumb.

### 25.4 Trapezoidal Integration-Tustin-Bilinear Method

If one uses the Trapezoidal Integral Approximation:

$$x(nT_s) - x((n-1)T_s) = \int_{(n-1)T_s}^{nT_s} \dot{x}(\tau) d\tau \approx \frac{T_s}{2} \left[ \dot{x}(nT_s) + \dot{x}((n-1)T_s) \right]$$

(25.30)

obtained by averaging the Forward and Backward Euler Integral Approximations, then one obtains

$$H_d(z) = H_c(s) \bigg|_{s = \frac{z - 1}{T_s(z + 1)}}$$

(25.31)

$$A_d = [I - A_cT_s]^{-1}[I + A_cT_s]$$

(25.32)

$$B_d = 2[I - A_cT_s]^{-1}B_c$$

(25.33)

$$C_d = \frac{T_s}{2}C_c[I - A_cT_s]^{-1}$$

(25.34)

$$D_d = D_c + \frac{T_s}{2}C_c[I - A_cT_s]^{-1}B_c.$$  

(25.35)

With this implementation, the left half $s$-plane gets mapped into unit circle in the $z$-plane. Hence, exponentially stable matrices $A_c$ get mapped into exponentially stable matrices $A_d$.

Bibliography


