Problem #2: Steady State Analysis, Laplace, Differential Equations

Consider the following causal LTI system:

\[ x(t) \rightarrow \frac{1}{s} \rightarrow \frac{1}{s+1} \rightarrow y(t) \]

a) Determine a differential equation which relates \( y \) to \( x \)

We begin in the s-domain. The transfer function from \( x \) to \( y \) is as follows:

\[
\frac{Y(s)}{X(s)} = \left( \frac{1}{s+1} \right) \frac{1}{s} = \frac{1}{s^2 + s}
\]

tf from \( x \) to \( y \)

Given this, we have \((s^2+s)Y(s) = X(s) \) or

\[ y + y = x \]

b) Suppose that \( x(t) = \cos(0.01t) \) \( g(t) \), \( y(0^-) = 0 \), \( y(0^-) = 1 \). Is the steady state output independent of the IC's? Determine the steady state output yss.

Because the above system has a pole at \( s=0 \), it is not stable. It is only marginally stable. Because of this, the effect of IC's do not decay away as \( t \to \infty \).

Given this, the steady state does depend on the IC's and hence we cannot use the method of the transfer function to compute the steady state output.

To determine the steady state output we examine the system's differential equation:

\[ y + y = x \]
Transforming the system's differential equation yields:

\[ [s^2Y(s) - sy(0) - y(0)] + [sY(s) - y(s)] = X(s) \]

or

\[ Y(s) = \frac{sy(0) + y(0) + \dot{y}(0)}{s(s+1)} + \left[ \frac{1}{s(s+1)} \right] X(s) \]

\[ \uparrow \quad \text{Term due to IC's} \]

\[ \uparrow \quad \text{Term due to forcing function} \]

\[ \text{exciting natural modes of system} \]

\[ \text{exciting natural modes} \]

Substituting in \( y(0) = 0, \dot{y}(0) = 1 \), and \( X(t) = \cos 0.01t + \varphi(t) \) (or \( X(s) = \frac{s}{s^2 + 10^{-4}} \)) yields

\[ Y(s) = \frac{1}{s(s+1)} + \left[ \frac{1}{s^2(s+1)} \right] \left[ \frac{s}{s^2 + 10^{-4}} \right] \]

\[ Y_1 + Y_2 \]

\[ = A + \frac{B}{s+1} + \frac{C}{s+1} + \frac{D}{s-j10^{-2}} + * \]

From this, it follows that

\[ y(t) = \left[ A + Be^{-t} + Ce^{-t} + 2D \cos(0.01t + LP) \right] \varphi(t) \]

\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]

\[ \text{marginally stable} \quad \text{stable} \quad \text{stable} \quad \text{marginally stable} \]

and hence the steady state output is given by

\[ yss(t) = A + 2D \cos(0.01t + LP) \]

where

\[ A = \lim_{s \to 0} sY_1(s) = \lim_{s \to 0} \frac{s}{s(s+1)} = 1 \]

\[ D = \lim_{s \to j10^{-2}} (s-j10^{-2}) Y_2(s) = \lim_{s \to j10^{-2}} \left[ \frac{1}{s^2(s+1)} \right] \frac{1}{s+j10^{-2}} \]

\[ = \frac{1}{j2(10^{-2})} = \frac{100}{2} e^{-j90^\circ} \]
From these calculations, it follows that

\[ yss(t) \approx 1 + 100 \cos(0.01t - 90^\circ) \]
\[ \approx 1 - 100 \sin 0.01t \]

**Note:** If we had used the method of the transfer function, then we would have obtained

\[ yss = H(j0.01) | \cos (0.01t + \arg H(j0.01)) \]

where \( H(s) = \frac{1}{s(s+1)} \) and hence

\[ H(j0.01) \approx \frac{1}{j0.01} = 100e^{-j90^\circ} \]

Given this, we would have

\[ yss \approx 100 \cos(0.01t - 90^\circ) \]
\[ \approx -100 \sin 0.01t \]

which we see is the contribution of the forcing function \( x(t) = \cos 0.01t + q(t) \) to the steady state output.

**Note:** While we see that the system \( H(s) = \frac{1}{s(s+1)} \)

integrates the input \( x(t) = \cos 0.01t + q(t) \) (since \( H(s) \approx \frac{1}{s} \) for small \( s \) like \( s=j0.01 \)), the system's I.C.'s affect the steady state output.