Problem 1 (Small Gain Theorem)
(a) Consider a dynamical system $T = \frac{1}{s^2 + 2s + 1}$ in a negative feedback loop with $\Delta = 0.5$. Is the closed loop system stable? Does the loop satisfy the conditions of the small gain theorem? Provide a visual pictorial. Is the small gain condition necessary for stability? Provide a (stable) $\Delta$ with $\|S\| = 0.5$ for all $\omega$ which destabilizes the feedback loop. Repeat this using the smallest possible (in magnitude) stable $\Delta$. In general, when is the small gain condition a necessary condition? State clearly when the small condition is necessary and sufficient for closed loop stability.

(b) Consider the plant $P = P_o e^{-s\delta}$ where $P_o = \frac{1}{s}$ and $\delta \geq 0$ is nominally zero. Suppose that the controller $K = 3 \left[ \frac{T + 2}{s} \right]$ is in a negative feedback loop with $P$. Determine the range for $\delta \in [0, \delta_{\text{max}}]$ such that the closed loop system is stable. Specify relevant imaginary closed loop poles. What range for $\delta$ does the small gain theorem yield for each of the uncertainty descriptions; i.e. multiplicative, additive, divisive, feedback? Specify all relevant frequencies. Provide relevant sketches. How does the small gain theorem result compare with the exact result obtained above?

(c) Consider the plant $P = P_o \left[ \frac{s^2}{s^2 + 2s + 1} \right]$ where $P_o = \frac{1}{s^2 + 1}$ and $\omega_n > 0$ is nominally infinite. Suppose that the controller $K = \frac{s}{2}$ is in a negative feedback loop with $P$. Determine the range for $\delta \in [\omega_n, \infty]$ such that the closed loop system is stable. Specify relevant imaginary closed loop poles. What range for $\delta$ does the small gain theorem yield for each of the uncertainty descriptions; i.e. multiplicative, additive, divisive, feedback? Specify all relevant frequencies. Provide relevant sketches. How does the small gain theorem result compare with the exact result obtained above?

Problem 2 (Model Based Compensator Design)
(a) Consider a two-input two-output plant $P = [A_p, B_p, C_p, D_p]$ with associated variables $u_p, x_p, y_p$. We wish to design a model based compensator for $P$. Suppose that an integrator is augmented in the second input channel (state $x_1$). Suppose that a double integrator is augmented in the first output channel (states: $x_2, x_3$). Also suppose that $\frac{1}{s^2 + 1}$ is augmented in the first output channel (states: $x_4, x_5$). Discuss how to determine a state space representation for the design plant (augmented system) with state $x = [x_p^T, x_1, x_2, x_3, x_4, x_5]^T$. Discuss the design process. Determine a state space representation $(A_k, B_k, C_k, D_k)$ for the final controller.

(b) Consider a plant $P = \left[ \begin{array}{c|c} 0 & \nu^T \\ \hline \nu & 1 \end{array} \right] = [A_p, B_p, C_p, D_p] = [1, 9, 1, -1]$ Show how to design a (model-based) control system such that the closed loop system (i) is stable, (ii) exhibits zero steady state error to step output disturbances, (iii) exhibits a settling time of approximately 5 sec, (iv) exhibits an overshoot of approximately 4.3%, (v) attenuates the impact of high frequency sensor noise on the controls. Support your design with a rough root locus plot. Just show how to set up the problem. You need not solve the equations.

Problem 3 (Performance and Stability Robustness)
Consider a SISO negative feedback loop with open loop transfer function $L$, $S \overset{\text{def}}{=} \frac{1}{1+e}$, $T \overset{\text{def}}{=} 1 - S$, $e = Sr$, and $y = Tr$.

(a) What does $S + T = 1$ tell us about command following, output disturbance attenuation, and noise attenuation at a particular frequency?

(b) Given upper bounds on the sensitivity and complementary sensitivity ($\|S\| < \alpha, \|T\| < \beta$), determine bounds on the upward gain margin $\gamma GM$, downward gain margin $\delta GM$, and phase margin $PM$. Are the derived bounds tight? Explain! Hint: Use Nyquist and inverse-Nyquist ideas.

(c) If $\alpha = 1$, what can be said about the upward gain margin and the open loop transfer function? If $\beta = 1$, what can be said about the downward gain margin and the open loop transfer function? In each case, what can be said about the phase margin? What can be said if $\alpha = 2? \beta = 2$?

(d) Given $\gamma GM$, $\delta GM$, and $PM$, determine lower bounds for $\|S\|_\infty$ and $\|T\|_\infty$. What if the open loop transfer function $L$ is known to possess pole and zero in the open right half plane at p and z, respectively? What can be said if p and z are close to one another? What if p = z?

(e) If $\gamma GM = 5$, what can be said about $\|S\|_\infty$ and $\|T\|_\infty$? What if $\delta GM = 0.2$? What if $PM = 30^\circ$?

(f) Is it possible to have great values for $\gamma GM$, $\delta GM$, $PM$, and still have a poor peak sensitivity? Explain.
(g) If \( L \) is non-minimum phase, what can be said about \( \|S\|_{\mathcal{H}^\infty} ? \) Suppose \( L \) possesses a time delay. What can be said about \( \|S\|_{\mathcal{H}^\infty} ? \) If \( L \) is unstable, what can be said about \( \|T\|_{\mathcal{H}^\infty} ? \) Explain. Let \( z, p > 0 \) denote an open loop right half plane zero and pole, respectively.

(h) Suppose that \( \|WS\|_{\mathcal{H}^\infty} \leq 1 \) for \( W = \left[ \begin{array}{c} s^2 + \omega_c M_s \\ s^2 + s + \omega_c p \end{array} \right] \) with \( \omega_c > 0 \), \( M_s \geq 1 \), \( \epsilon \geq 0 \), and \( \omega_p = f z > M_s^2 \omega_c \). Sketch \( Ws^{-1} \). What does it represent? Relate \( \omega_c \) to \( z \). What can be said for \( \epsilon = 0 \), \( M_s = 2 \), and \( f = \frac{1}{2}, 1, 2, \infty ? \) Physically, what can the parameter \( f \) be thought of as representing? Now suppose \( f = \infty \) and \( \epsilon > 0 \). Relate \( M_s \) to \( z, \omega_c, \epsilon \). When does \( M_s \to \infty \)? Hint: As \( \omega_c \) approaches what?

Now suppose that \( \|WT\|_{\mathcal{H}^\infty} \leq 1 \) for \( W = \left[ \begin{array}{c} s^2 + \omega_c M_t \\ s^2 + s + \omega_c p \end{array} \right] \) with \( \omega_t > 0 \), \( M_t \geq 1 \), \( \epsilon \geq 0 \), and \( \omega_p = f p < \omega_t / M_t^2 \). Sketch \( W_{s^{-1}} \). What does it represent? Relate \( \omega_t \) to \( p \). What can be said for \( \epsilon = 0 \), \( M_t = 2 \), and \( f = 0, \frac{1}{2}, 1, 2 \)? Physically, what can the parameter \( f \) be thought of as representing? Now suppose \( f = 0 \) and \( \epsilon > 0 \). Relate \( M_t \) to \( p, \omega_t, \epsilon \). When does \( M_t \to \infty \)? Hint: As \( \omega_t \) approaches what?

(i) Give necessary and sufficient conditions on \( |L| \) and \( \angle L \) so that \( |S| < \alpha \) (for all \( \omega \)). Now let \( \xi \) fix.

- Give conditions on \( |L| \) such that \( |S| < \alpha \) independent of \( \angle L \). Explain your result with a clear picture?
- Now suppose that the above magnitude conditions on \( |L| \) are not satisfied. Give conditions on \( \angle L \) such that \( |S| < \alpha \) for any \( |L| \)? Explain your result with a clear picture?

Now suppose that \( \angle L \) does not satisfy the angle condition just obtained. Give conditions on \( |L| \) (that depend on \( \angle L \)) for \( |S| < \alpha \). Explain your result with a clear picture?

**Problem 4 (LQR and LQ Servo)**

(a) Consider \( P = \frac{\beta}{s-\alpha} \). Show how to design an LQ servo that will guarantee zero steady state error to step commands. Clearly explain the process. Hint: \( J = \int_0^\infty (y^2 + \rho u^2) \, dt \), \( y = Mx \). Suppose \( a = b = 1 \). Determine \( \rho \) such that the LQ loop downward gain margin is \( \gamma_{GM} = \frac{1}{2} \). Determine \( G_{LQ} \), upward gain margin \( \gamma_{GM} \), \( \omega_g \), \( PM \), and \( DM \). Clearly draw your final LQ servo architecture. What would you expect to happen to the design’s properties as \( \rho \to \infty ? \) \( \rho \to 0 ? \) Explain. If a PI controller was desired in the final architecture, how would one modify the design process?

(b) Can \( G = \frac{\alpha}{\beta} \) be an LQ loop? Explain.

(c) Can \( G = \frac{\beta}{\alpha} \) be an LQ loop? Explain.

(d) Can \( G = \frac{\alpha}{\beta} \) be an LQ loop? Explain.

(e) Can \( G = \frac{\alpha}{\beta s + \beta + \alpha} \) be an LQ loop? Explain.

(f) Suppose \( G_{OL} = \frac{\alpha s + \beta}{\alpha (s + 1)} \) where \( g, z > 0 \). What can be said as \( \rho \to 0, \infty \)? Sketch a root square locus.

(g) Suppose \( G_{OL} = \frac{\alpha s + \beta}{\alpha (s + 1)} \) where \( g, z > 0 \). What can be said as \( \rho \to 0, \infty \)? Sketch a root square locus.

**Problem 5 (Kalman Filter)**

(a) Consider \( P = \frac{\beta}{s-\alpha} \). Show how to design a Kalman filter for this system. Clearly explain the process. Hint: LQR solution can be applied to dual system. Compute \( \Sigma_{\mu} \), \( \min E(||x - \hat{x}||^2) \), \( H_{\mu} \). For \( z = 1 \) and \( \mu = 1 \), compute \( \Sigma, H, G_{KF} \), upward gain margin \( \gamma_{GM} \), downward gain margin \( \delta_{GM} \), \( \omega_g, PM, \) and \( DM \). What would you expect to happen to the design’s properties as \( \mu \to \infty \)? \( \mu \to 0 ? \) Explain. Draw a clear (detailed, insightful) block diagram for your Kalman filter.

(b) Can \( G = \frac{\alpha s + \beta}{\alpha (s + 1)} \) be a KF loop? Explain.

(d) Can \( T = \frac{\alpha s + \beta}{s + \beta + \alpha} \) be a KF loop? Explain.

(e) Can \( S = \frac{\alpha s + \beta}{s + \beta + \alpha} \) be an S_{KF}? Explain.

(f) Suppose \( G_{OL} = \frac{\alpha s + \beta}{\alpha (s + 1)} \) where \( g, z > 0 \). What can be said as \( \mu \to 0, \infty \)? Sketch a root square locus.

(g) Suppose \( G_{OL} = \frac{\alpha s + \beta}{\alpha (s + 1)} \) where \( g, z > 0 \). What can be said as \( \mu \to 0, \infty \)? Sketch a root square locus.
Problem 6 (LQG and LTR)
(a) Describe LQG/LTR at output (LTRO)?
(b) Describe LQG/LTR at input (LTRI)? Prove it.
(c) Consider \( P = \frac{1}{z} \left[ \frac{z-1}{s-1} \right] \) for \( z = 10 \). Show how to design an LQG/LTRO controller such that the closed loop system exhibits (1) zero steady state error to step input disturbances \( d_i \), (2) a unity gain crossover frequency \( \omega_g \) of 3 rad/sec at the output. Describe the entire design process. Show (symbolically) the equations that need to be solved. What dynamic augmentation is required? Is it essential to perform the dynamic augmentation at the input? Explain.
(b) Give sufficient conditions for LTRO. Is this condition necessary? Assuming that LTR is possible, give a block diagram proof for LTR.

Problem 7 (Weighted \( H^\infty \) Sensitivity Optimization)
Describe the weighted \( H^\infty \) optimization problem. Specify the generalized plant in terms of the plant \( P \) and the weightings. Give a state space representation for \( G \). Show how to design a controller for \( P = \frac{1}{s-1} \) such that the following design specifications are satisfied: (1) closed loop system is stable, (2) closed loop system exhibits zero steady state error to step reference commands, (3) unity gain crossover is near 3 rad/sec, (4) \( |L(j\omega)| \approx 200 \) (46 dB), (5) \( |L(j\omega)| \approx 0.1 \) (-20 dB), (6) settling time of about 5 sec for step reference commands, (7) About 5% overshoot results for step reference commands. Specify all weights and parameters. Specify the final controller transfer function.

Problem 8 (Analysis of a Closed Loop System)
(a) Consider the control system design for the PUMA robotic manipulator (LQG/LTRO). Discuss the singular values of \( L_u \). Discuss the singular values of \( T_{d,u} \). Show how to construct the worst case input disturbance. In what sense is it the worst case input disturbance?
(b) Consider the F8 aircraft flight control system (LQ servo). Discuss the singular values of \( L_e \), \( S_e \) and \( T_e \). How are the properties at the error different from those at the controls? Be very specific.