Problem 1 (Small Gain Theorem)

(a) Consider a dynamical system $T = \frac{b}{s^2+a}$ in a negative feedback loop with $\Delta = c$ where $b > a > 0$. Suppose that $\frac{b}{a} > 1$. Is the closed loop system stable? Does the loop satisfy the conditions of the small gain theorem? Provide a visual pictorial. Is the small gain condition necessary for stability? State clearly when the small condition is necessary and sufficient for closed loop stability.

(b) Consider the plant $P = P_o \left[ \frac{s-\alpha}{s+\omega_s} \right]$ where $P_o = \frac{1}{s-1}$ and $\alpha > 0$. Suppose that the controller $K = \frac{2s+1}{s}$ is in a negative feedback loop with $P$. Determine the range for $\alpha$ such that the closed loop system is stable. Specify all relevant imaginary closed loop poles. What range for $\omega_s$ does the small gain theorem yield for each of the uncertainty descriptions; i.e. multiplicative, additive, divisive, feedback? Specify all relevant frequencies. How does the small gain theorem result compare with the exact result obtained above?

Problem 2 (Model Based Compensator Design)

(a) Consider a two-input two-output plant $P = [A_p, B_p, C_p, D_p]$ with associated variables $u_p, x_p, y_p$. We wish to design a model based compensator for $P$. Suppose that an integrator is augmented in the first output channel (state $x_1$). Suppose that a double integrator is augmented in the second output channel (states: $x_2, x_3$). Also suppose that $\frac{1}{s^2+\omega_s^2}$ is augmented in the first input channel (states: $x_4, x_5$). Determine a state space representation for the design plant (augmented system) with state $x = [x_p^T, x_1, x_2, x_3, x_4, x_5]^T$. Discuss the design process. Give a state space representation $(A_k, B_k, C_k, D_k)$ for the final controller.

(b) Consider a plant $P = \frac{W-1}{s^2+\omega_s^2} = [A_p, B_p, C_p, D_p] = [1, 9, 1, -1]$ Design a (model-based) control system such that the closed loop system (a) is stable, (b) exhibits zero steady state error to step output disturbances, (c) exhibits a settling time of approximately 5 sec, (d) exhibits an overshoot of approximately 4.3%, (e) attenuates the impact of high frequency sensor noise on the controls. Support your design with a rough root locus plot.

Problem 3 (Performance and Stability Robustness)

Consider a SISO negative feedback loop with open loop transfer function $L$, $S \stackrel{\text{def}}{=} \frac{1}{1+L}$, $T \stackrel{\text{def}}{=} 1-S$, $e = Sr$, and $y = Tr$.

(a) What does $S + T = 1$ tell us about command following, output disturbance attenuation, and noise attenuation at a particular frequency?

(b) Given upper bounds on the sensitivity and complementary sensitivity ($\|S\| < \alpha$, $\|T\| < \beta$), determine bounds on the upward gain margin $\Upsilon GM$, downward gain margin $\Upsilon GM$, and phase margin $PM$. Are the derived bounds tight? Explain! Hint: Use Nyquist and inverse-Nyquist ideas.

(c) If $\alpha = 1$, what can be said about the upward gain margin and the open loop transfer function? If $\beta = 1$, what can be said about the downward gain margin and the open loop transfer function? In each case, what can be said about the phase margin?

(d) Given $\Upsilon GM$, $\Upsilon GM$, and $PM$, determine lower bounds for $\|S\|_{\mathcal{H}_\infty}$ and $\|T\|_{\mathcal{H}_\infty}$.

(e) If $\Upsilon GM = 10$, what can be said about $\|S\|_{\mathcal{H}_\infty}$ and $\|T\|_{\mathcal{H}_\infty}$? What if $\Upsilon GM = 0.1$? What if $PM = 45^\circ$?

(f) Is it possible to have great values for $\Upsilon GM$, $\Upsilon GM$, $PM$, and still have a poor peak sensitivity? Explain.

(g) If $L$ is non-minimum phase, what can be said about $\|S\|_{\mathcal{H}_\infty}$? If $L$ is unstable, what can be said about $\|T\|_{\mathcal{H}_\infty}$? Explain.

Let $z, p > 0$ denote an open loop RHP zero and pole, respectively.

(h) What can be said about $\min\{\|S\|_{\mathcal{H}_\infty}, \|T\|_{\mathcal{H}_\infty}\}$? What can be said if $p$ and $z$ are close to one another?

(i) Suppose that $\|WS\|_{\mathcal{H}_\infty} \leq 1$ for $W = \frac{z+\omega_sM_s}{\omega_s}$ with $\omega_s > 0$, $M_s \geq 1$. Relate $\omega_s$ to $z$. What can be said when $M_s = 1, 2, \infty$? Now suppose that $\|WT\|_{\mathcal{H}_\infty} \leq 1$ for $W = \frac{s+\omega_tM_t}{\omega_t}$ with $\omega_t > 0$, $M_t \geq 1$. Relate $\omega_t$ to $p$. What can be said when $M_t = 1, 2, \infty$? What do these relationships tell us about control system bandwidth?

Problem 4 (LQ Servo)
Consider $P = \frac{b}{s-a}$. Show how to design an LQ servo that will guarantee zero steady state error to step commands. Clearly explain the process. Hint: $J = \int_0^\infty (y^2 + \rho u^2) \, dt$, $y = Mx$. Suppose $a = b = 1$. Determine $\rho$ such that the LQ loop downward gain margin is $\downarrow GM = \frac{1}{3}$. Determine $G_{LQ}$, upward gain margin $\uparrow GM$, $\omega_g$, $PM$, and $DM$. Clearly draw your final LQ servo architecture. What would you expect to happen to the design’s properties as $\rho \to \infty$? $\rho \to 0$? Explain. If a PI controller was desired in the final architecture, how would one modify the design process?

**Problem 5 (Kalman Filter)**

Consider $P = \frac{z}{s^2}$. Show how to design a Kalman filter for this system. Clearly explain the process. Hint: LQR solution can be applied to dual system. Compute $\Sigma_\mu$, $\min E(||x - \hat{x}||^2)$, $H_\mu$. For $z = 1$ and $\mu = 1$, compute $\Sigma$, $H$, $G_{KF}$, upward gain margin $\uparrow GM$, downward gain margin $\downarrow GM$, $\omega_g$, $PM$, and $DM$. What would you expect to happen to the design’s properties as $\mu \to \infty$? $\mu \to 0$? Explain.

**Problem 6 (LQG and LTR)**

(a) Consider $P = \frac{1}{s^2 + \frac{z}{s+1}}$ for $z = 10, 4, 2$. Show how to design an LQG/LTRO controller such that the closed loop system exhibits (1) zero steady state error to step input disturbances $d_i$, (2) a unity gain crossover frequency $\omega_g$ of 3 rad/sec at the output. Describe the entire design process. Show (symbolically) the equations that need to be solved. What dynamic augmentation is required? Is it essential to perform the dynamic augmentation at the input? Explain. For each $z$, specify your selected $\mu$, target loop margins, final $\rho$, final achieved margins. Discuss how well the target loop is recovered for each $z$.

(b) Give sufficient conditions for LTRO. Is this condition necessary? Assuming that LTR is possible, give a block diagram proof for LTR.