State Feedback (Full State Feedback)
\[ \dot{x} = Ax + Bu \]

Assumption: \( x \) is known and available for feedback. Consider the following state feedback control law.

\[
\begin{array}{c}
\text{control} \\
\text{Gain} \text{ matrix}
\end{array}
\begin{array}{c}
\text{state} \\
\end{array}
\]

\[ u = -Gx \]

Dimension of \( G = m \times n \)

* How do we select ("design") nice \( G \)'s?

- If the system \((A, B)\) is controllable, you can design a nice \( G \).

- If the system is controllable, you can make your open loop system cook eggs!

  - Mathematical statement, not Engineering Statement
  - Theories have limitations, Stupidity does not.

\[ \dot{x} = (A - BG)x \]

- New \text{A matrix}
- Closed loop \text{A matrix}

We want \( G \) to be selected such that \( A - BG \) is stable!
**Example**

\[
\begin{align*}
\dot{x}_1 &= x_1 + x_2 \\
\dot{x}_2 &= u
\end{align*}
\]

Using state feedback,

\[
u = -Gx = -\begin{bmatrix} g_1 & g_2 \end{bmatrix} x
\]

\[
\dot{x} = (A - BG)x
\]

\[
A - BG = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -g_1 & -g_2 \end{bmatrix}
\]

What is closed loop characteristic equation?

\[
\det(sI - A + BG) = 0
\]

We are finding the eigenvalues of \(A - BG\)

\[
\det \begin{bmatrix} s - 1 & -1 \\ g_1 & s + g_2 \end{bmatrix} = 0
\]

\[
s^2 + (g_2 - 1)s - g_2 + g_1 = 0
\]
\[ s^2 + (g_2 - 1)s - g_2 + g_1 = 0 \]

Suppose \( \phi_c = s^2 + 2s + 2 \)

Choose \[
\begin{array}{l}
g_2 = 3 \\
g_1 = 5
\end{array}
\]

shows how we can use state feedback to get poles at desirable locations.

- we have been able to stabilize the system !!

- why could we stabilize the system using state feedback ?
  \( \rightarrow \) because the system \((A, B)\) is controllable !!

* **Pole Placement Fact**

Given an Open Loop System \((A, B)\)

\[
x = Ax + Bu
\]

one can use state feedback

\[ u = -Cx \]

so the closed loop system

\[
\dot{x} = (A - B6)x
\]

has arbitrary closed loop Poles (eigenvalues)

iff \((A, B)\) is controllable.

If the open loop system is controllable, you can make it cook eggs!

---

official technical term in literature!
Example

\[ \dot{x}_1 = x_1 + v \]
\[ \dot{x}_2 = 0x_2 - u \]
\[ v = x_2 + u \]

Pole-zero cancellation at system input.

s-1 mode is not controllable

Form \((A, B)\) let \(u = -Gx\)

→ can you place both poles at \(s = -1, -2\)

or

→ can we use state feedback to move the open loop poles to these desirable locations?

HELL NO !!!

One of the closed loop poles is at \(s = 1\)

→ If a certain mode is uncontrollable, that mode will not move!

\[ \dot{x}_1 = x_1 + x_2 + u \]
\[ \dot{x}_2 = -u \]

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \]

\[ G[A, B] = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \Rightarrow \text{NOT CONTROLLABLE} ! \]
\[ u = -[g_1, g_2] x \]

\[ A - BG = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} [g_1, g_2] = \begin{bmatrix} 1 - g_1 & 1 - g_2 \\ 0 & 0 \end{bmatrix} \]

\[ \det(sI - A + BG) = \det \begin{bmatrix} s - 1 + g_1 & -1 + g_2 \\ -g_1 & s - g_2 \end{bmatrix} = s^2 + (-g_2 - 1 + g_1)s + g_2(1 - g_1) + g_1(g_2 - 1) \]

Claim: \( s = 1 \) is a root of this regardless of \( g_1, g_2 \) selected!

The closed loop system will have a pole at \( s = 1 \) (not movable)

check

\[ 1 + (-g_2 - 1 + g_1) + g_2(1 - g_1) + g_1(g_2 - 1) = 0 \]

uncontrollable modes cannot be moved.

Pole Placement Fact

uncontrollable modes cannot be moved at all using constant gain full state feedback!
Linear systems

Pole Placement

$\rightarrow$ Bass-Gura Formula
$\rightarrow$
$\rightarrow$

useless for multivariable design.

Placing Poles is not enough

--- Remember that we must also be robust to uncertainty.

sensitivity $\rightarrow$ want small bumps $\rightarrow$ comp. sensitivity

$\rightarrow$ to be robust to uncertainty

--- How do we design C such that

$\rightarrow$ A-BG is stable
$\rightarrow$ Closed Loop System is "Robust"

Initial Answer: Linear Quadratic Regulator Problem (LQR)

$\Rightarrow$ minimize quadratic cost function

$$J(u) = \frac{1}{2} \int_{0}^{\infty} x^T Q x + u^T R u \, dt$$

subject to constraint

$$\dot{x} = Ax + Bu$$

A, R are design parameters that we pick
Answer

\[ u = -Gx \]
\[ G = R^{-1}B^TK \]

\[ K \] - solution to a Control Riccati Algebraic Equation (CARE)

\[ \uparrow \]

in MATLAB