Problem 1 (SSR for Inner-Outer Loop Feedback Structure)
Sketch a block diagram for the following system. Also, determine a state space representation from the external signals \((r, d_u, d_y, n_f, n_y)\) to the internal signals \((y, u, u_y, u_k, u_f, e)\).

\[
\begin{align*}
\dot{x}_p &= A_p x_p + B_p u_p \quad u_p = d_u + u && u = u_k - u_f \quad u_f = C_f x_f \quad \dot{x}_f = A_f x_f + B_f x_f \quad y = C_p x_p + d_y \\
x_{fb} &= C_{fb} x_p + n_f \quad u_k = C_k x_k \quad \dot{x}_k = A_k x_k + B_k e \quad e = r - y - n_y 
\end{align*}
\] (96) (97)

Problem 2 (SVD)
(a) Consider the non-singular linear matrix transformation \(y = Mx\), where \(M\) is a \(2 \times 2\) real matrix. Provide a clearly labelled pictorial which describes how the matrix \(M\) maps vectors \(x\) into vectors \(y\). Use gain concepts to discuss what is meant by certain input directions being maximally amplified and others being minimally amplified.

(b) Consider a MIMO LTI system \(T_{du}\), where \(T_{du}(j\omega) = U \Sigma \Sigma^H\) is an svd, \(\|T_{du}\|_{H^\infty} = \sigma_{\max} [T_{du}(j\omega_0)] = \sigma_1\), \(U = \begin{bmatrix} u_{11} & e_j \lambda_{u11} \\
u_{12} & e_j \lambda_{u12} \end{bmatrix}, \Sigma = \text{diag}(\sigma_1, \sigma_2), V = \begin{bmatrix} v_{11} & e_j \lambda_{v11} \\
v_{12} & e_j \lambda_{v12} \end{bmatrix}, \text{ and } \sigma_1 > \sigma_2\). Show how to construct the “maximally-amplified” and “minimally-amplified” sinusoidal disturbances \(d\) and their associated steady state outputs \(y\). In what sense are the disturbances maximally and minimally amplified? How is the energy of the output \(\|y\|_{L^2}\) related to the energy of the input \(\|d\|_{L^2}\)?

(c) Construct a constant matrix \(\Delta\) with minimal \(\sigma_{\max} |\Delta|\) such that the matrix \(I - T_{du}(j\omega_0)\Delta\) is singular.

Problem 3 (Small Gain Theorem, Stability Robustness Tests for Unstructured Uncertainty)
(a) Consider a dynamical system \(T = \frac{\delta}{s+1}\) in a positive feedback loop with an uncertainty \(\Delta\). State sufficient “small-gain” conditions for closed loop stability. Now suppose that \(\Delta = -\frac{1}{10}\). Does the loop satisfy the conditions of the small gain theorem? Provide a visual pictorial. What does this tell us about the stability of the closed loop system? Are the small gain conditions necessary? When are they necessary? Be very precise.

(b) Does there exist a \(\Delta\) with \(|\Delta(j\omega)| = \frac{1}{10}\) which will destabilize the feedback loop? If so, find one. Repeat this for \(|\Delta(j\omega)| = \frac{1}{10}\) and then \(|\Delta(j\omega)| = 2\).

(c) Consider the uncertain MIMO dynamical systems

(i) \(P = P_0 + W_1 \Delta_1 W_2\) 
(ii) \(P = [I + W_1 \Delta_2 W_2] P_0\) 
(iii) \(P = P_0 [I - W_1 \Delta_2 W_2]^{-1}\) 
(iv) \(P = [I - P_0 W_1 \Delta_2 W_2]^{-1} P_0\)

where \(P_0\) represents a nominal MIMO LTI plant model, \(\Delta_i\) represents a stable LTI uncertainty satisfying \(\sigma_{\max} |\Delta_i(j\omega)| \leq 1\) for all \(\omega\), and \((W_1, W_2)\) represent stable LTI weighting functions. Consider a negative feedback system with controller \(K\) and plant \(P\). For each of the above uncertainty descriptions, determine sufficient conditions on \(P_0\) and \(K\) such that the feedback loop is stable for all \(\Delta\) satisfying \(\sigma_{\max} |\Delta(j\omega)| \leq 1\) for all \(\omega\). Are the conditions necessary? Explain clearly the implication if one of the conditions is violated.

Problem 4 (Stability Robustness, Small Gain Concepts)
For each of the following determine exact conditions for stability. Also determine approximate sufficient conditions via small gain concepts.

(A) Time Delay. Consider the LTI plant \(P = P_0 e^{-\delta s}\) where \(P_0 = \frac{1}{s+1}, p > 0, \text{ and } \delta > 0\). Consider the nominal controller \(K = k > p\) in a classic negative feedback configuration.
(a) Determine the delay margin of the nominal design; i.e. \(L_0 = P_0 K = \frac{k}{s+1}\).

(b) Now we wish to use small gain theorem ideas to estimate the delay margin. Do so by considering a multiplicative modeling error characterization. Comment on the conservatism of the answer obtained in (b) with respect to the exact answer obtained in (a).

(B) RHP Zero. Now consider the LTI plant \(P = P_0 \left( \frac{z+\delta}{z} \right)\) where \(P_0 = \frac{1}{s+1}, z > 0, \text{ and nominal controller } K = \frac{z}{s+1}\). Consider a class negative feedback configuration. (a) Show that the closed loop system is stable when \(z\) is sufficiently large. Provide a supporting root locus plot. What is the minimum value of \(z, z_{\min}\), for which closed loop stability is maintained?

(b) Now we wish to use small gain theorem ideas to estimate \(z_{\min}\). Do so by considering suitable modeling error characterizations; e.g. additive, multiplicative, divisive, feedback. Clearly explain which characterisation
works and which does not. For suitable characterizations, provide a supporting Bode asymptotic magnitude plot illustrating your small gain test. Comment on the conservatism of the (approximate) answers obtained in (b) with respect to the exact answer obtained in (a). When the plant has a RHP zero, what general conclusion can be made about suitable modeling error characterizations?

(C) High Frequency Actuator Mode. Now consider the LTI plant \( P = P_o \frac{s + a}{s + 2} \) where \( P_o = \frac{1}{s + 2}, a \geq 0, \) and nominal controller \( K = \frac{2}{s} \) in a classic negative feedback configuration. (a) Show that the closed loop system is stable when \( a \) is sufficiently large. What is the minimum value of \( a, a_{min}, \) for which closed loop stability is maintained? Hint: Write the closed loop characteristic equation in root locus form with respect to the parameter \( a; i.e. \Phi(s,a) = p_1(s) + a p_2(s) \) where \( p_1 \) and \( p_2 \) are polynomials. Examine the corresponding root locus with respect to \( a \) to determine \( a_{min}. \) (b) Now we wish to use small gain theorem ideas to estimate \( a_{min}. \) Do so by considering suitable modeling error characterizations; e.g. additive, multiplicative, divisive, feedback. Clearly explain which characterization works and which does not. For suitable characterizations, provide a supporting Bode asymptotic magnitude plot illustrating your small gain test. Comment on the conservatism of the (approximate) answers obtained in (b) with respect to the exact answer obtained in (a).

Problem 5 (Robust Stability, SSV, and Robust Performance for SISO Plants)
Consider the SISO LTI nominal plant \( P_o, \) associated nominal controller \( K, \) nominal sensitivity \( S_o = 1 + P_o K, \) and nominal complementary sensitivity \( T_o = 1 - S_o. \) Suppose that the actual plant is \( P = \left[ 1 + W_2 \Delta_m P_o \right] \frac{1}{1 - W_1 \Delta_d} \) where \( \Delta_m \) denotes a stable LTI multiplicative modeling uncertainty, \( \Delta_d \) denotes a stable LTI divisive modeling uncertainty, \( |\Delta_m(j\omega)| \leq 1, |\Delta_d(j\omega)| \leq 1, \) and \( W_i (i = 1,2) \) is a stable LTI weighting function. Derive necessary and sufficient conditions for the nominal closed loop system to be robust with respect to the above uncertainty. Show how this is a structured singular value problem. Show how this problem is related to an equivalent robust performance problem.

Problem 6 (Margin-Sensitivity Bounds, Robustness, Push-Pop Effects, Bandwidth Constraints)
(a) Margin Bounds. Given upper bounds on the sensitivity and complementary sensitivity \( |S| < \alpha, |T| < \beta, \) determine bounds on the upward gain margin \( \uparrow GM, \) downward gain margin \( \downarrow GM, \) and phase margin \( PM. \) If \( \alpha = 1, \) what can be said about the upward gain margin, downward gain margin, phase margin, and the open loop transfer function? What if \( \beta = 1? \) What if \( \alpha = \beta = 2? \)
Suppose \( L \) is the open loop transfer function. If \( L \) is non-minimum phase, what can be said about \( \|S\|_{\infty}? \) If \( L \) is unstable, what can be said about \( \|T\|_{\infty}? \)
(b) Robustness. Do good classic robustness margins ensure robustness? Explain. Provide a SISO Nyquist illustration. Also provide a two-input two-output example illustrating that per channel margins are not sufficient for robustness.
(c) Sensitivity Bounds. Given the upward gain margin \( \uparrow GM, \) downward gain margin \( \downarrow GM, \) and phase margin \( PM, \) determine lower bounds on \( \|S\|_{\infty}, \) and \( \|T\|_{\infty}. \) Given that \( p \) and \( z \) represent an open loop pole and zero, give bounds on \( \|S\|_{\infty} \) and \( \|T\|_{\infty} \).
(d) Sensitivity Push-Pop Effects. What are desirable feedback properties? Explain sensitivity push-pop effects for bandwidth constrained feedback systems. Provide supporting formulae and graphical illustrations.
(e) Bandwidth Constraints. Explain how right half plane poles and zeros place constraints on feedback systems. Provide supporting formulae and graphical illustrations.

Problem 7 (LQR)
(a) Show how to design an LQR-servo for the plant \( P = \frac{z^2 - 2}{s - p} = [0, 1, z, -1]. \) Use \( R = \rho > 0. \) Integral action is desired so that step reference commands are followed. Is integral augmentation necessary for this? Hint: Sketch the structure of the final LQR-based feedback loop. Set up and solve the CARE for \( K_p. \) Compute the control gain matrix \( G_p. \)
(b) Describe guaranteed properties at controls. Are their guaranteed properties at error? Discuss. Given that input disturbances occur at the plant input, will this structure reject step input disturbances. Give a clear explanation.

Problem 8 (Kalman Filter: State Estimator)
(a) Show how to design a Kalman filter for \( P = \frac{z^2 - 2}{s - p} = [A, B, C, D] = [p, 1, z - p, -1]. \)
Assume \( L = B = 1, \Xi = 1, \) and \( \Theta = \mu. \)
(b) What are the associated estimator equations?
(c) Sketch a block diagram for the estimator.
(d) Determine solution to FARE.
(e) Determine \( H \) and \( G_{KF}. \)
Problem 9 (LQG, LTR and Loop Shaping)
Summarize the LQG solution. What closed loop properties do LQG controllers offer? Do LQG controllers generally enjoy the stability robustness properties associated with LQR and KF loops? Describe the LQG/LTRO procedure. Give a proof for LTRO. Give sufficient conditions for LTRO. Are the conditions necessary?
(b) Discuss how the (target) KF loop can be shaped. Specifically, discuss how to match the singular values at (i) low frequencies, (ii) high frequencies, and (iii) all frequencies (Explain).

Problem 10 (Forming Generalized Plant, Weighted $\mathcal{H}^\infty$ Mixed-Sensitivity Minimization)
(a) Consider a nominal plant $P = \frac{2}{s-1}$ with state $x_p$ with weightings $W_1 = k_1 \left[ \frac{s+2}{s+3} \right] = k_1 + \frac{k_1 (s+1)}{s+3}$ with state $x_1$ and state space $A_1 = -p_1$, $B_2 = z_1 - p_1$, $C_1 = D_1 = k_1$, $W_2 = k_2$, $W_3 = k_3 (s + z_2)$. Sketch a block diagram of the overall closed loop system. Determine the transfer function matrix for the generalized plant. Let $x = [x_p \ x_1]^T$ represent the state of the closed loop system. Determine the state space representation for the generalized plant.
(b) Given $P = \frac{1}{s-1}$, use computer to determine weightings $W_1$, $W_2$, $W_3$ which yield an $\mathcal{H}^\infty$ mixed-sensitivity controller that approximates $K = 9 \left[ \frac{s+2}{s} \right]$. Note that dominant closed loop poles are at $s = -4 \pm j3$. Describe the design procedure. How did you select the weightings?