(1) **Feedback System SSR.** Consider a classical negative feedback loop with plant \( P = [A_p, B_p, C_p] \) and controller \( K = [A_k, B_k, C_k] \). Determine a state space representation for the map from the exogenous signals \((r, d_i, d_o, n)\) to the internal signals \((y, u, e)\).

(2) **Dynamic Augmentation.** Consider a two-input four-output plant \( P = [A_p, B_p, C_p, D_p] \) with input \( u_p \), state \( x_p \), and output \( y_p \). Suppose that the above system is augmented with

- \( \frac{s(s+a)}{s} \) (input: \( u_1 \); state: \( x_1 \)) in the first input channel,
- \( \frac{1}{s} \) (state: \( x_2 \); output: \( y_1 \)) in the first output channel, and
- \( \frac{\omega^2}{s^2 + \omega^2} \) (states: \( x_3, x_4 \); output: \( y_2 \)) in the second output channel.

Determine a state space representation for the augmented two-input four-output system with \( u \) denoting the new input, \( x = [x_p^T \quad x_1 \quad x_2 \quad x_3 \quad x_4]^T \) the state, and \( y \) the output of the augmented system.

(3) **Coordinate Transformation: Change of Units.** Consider a MIMO plant \( P = [A_1, B_1, C_1, D_1] \) with input \( u_1 \), internal state \( x_1 \), and output \( y_1 \). Suppose that we wish to change the units of \( u_1, x_1, \) and \( y_1 \). Show how this can be done. Let \( u_2, x_2, \) and \( y_2 \) denote the new input, state, and output vectors, respectively. Relate the new transfer function matrix to the original transfer function matrix. Discuss what fundamental dynamical system properties are invariant under the above transformation. Discuss what changes.

(4) **Stabilizability, State Feedback, Pole-Placement.** Consider the dynamical system

\[
\begin{align*}
\dot{x}_1 &= p_1 x_1 + (p_2 - p_1) x_2 + ku \\
\dot{x}_2 &= p_2 x_2 + ku \\
y &= gx_1
\end{align*}
\]

with \( p_1, p_2 > 0 \). Is the above system stabilizable? Explain. Discuss what pole-placement attributes can be achieved via constant gain full state feedback; i.e. where can the closed loop poles be placed. Under what conditions on \( p_1, p_2 \) would the system be stabilizable?

(5) **Model Based Compensator.** Consider the plant \( P = 1 = [A_p, B_p, C_p, D_p] \) with \( A_p = 0, B_p = 0, C_p = 0, D_p = 1 \). Design a model based compensator which guarantees closed loop stability, zero steady state error to sinusoidal input disturbances with frequency \( \omega_o = 10 \), and achieves dominant closed loop poles at \( s = -2 \pm j1 \) and \( s = -20 \pm j15 \). Hint: Order the states of your augmented system (i.e. design plant) as \( x = [x_p^T \quad x_1 \quad x_2]^T \). NOTE: You only need to set up the equations.
(6) SVD. (a) Consider the non-singular linear matrix transformation \( y = Mx \) where \( M \) is a \( 2 \times 2 \) real matrix. Provide a carefully labelled pictorial which describes how the matrix \( M \) maps vectors \( x \) into vectors \( y \). Use gain concepts to discuss what is meant by certain input directions being maximally amplified and others being minimally amplified.

(b) Now consider the two-input two-output (TITO) transfer function matrix \( T_{dy} \) from torque disturbances \( d = [d_1 \ d_2]^T \) to yaw-roll attitude angles \( y = [y_1 \ y_2]^T \) for a geosynchronous satellite. Suppose that a singular value decomposition at \( \omega_o \) rad/sec results in the following directionality information:

\[
\begin{align*}
v_1 &= \begin{bmatrix} v_{11} e^{i\phi_{11}} & v_{12} e^{i\phi_{12}} \end{bmatrix}^T, \quad \sigma_1 = \|T_{dy}\|_{\mathcal{H}^\infty}, \quad u_1 = \begin{bmatrix} u_{11} e^{i\phi_{11}} & u_{12} e^{i\phi_{12}} \end{bmatrix}^T.
\vspace{0.5em}\n
v_2 &= \begin{bmatrix} v_{21} e^{i\phi_{21}} & v_{22} e^{i\phi_{22}} \end{bmatrix}^T, \quad \sigma_2, \quad u_2 = \begin{bmatrix} u_{21} e^{i\phi_{21}} & u_{22} e^{i\phi_{22}} \end{bmatrix}^T.
\end{align*}
\]

What is the associated worst-case sinusoidal disturbance? What is the associated steady state output? In what sense is the worst case disturbance amplified? How is the energy of the output \( \|y\|_{\mathcal{L}^2} \) related to the energy of the input \( \|d\|_{\mathcal{L}^2} \)?

(c) Describe how you might use \( \mathcal{H}^\infty \) optimal control theory to directly improve the above disturbance response. What would you weight? How might you select the weighting function?

(7) Small Gain Theorem and Dynamic Uncertainty. (a) Consider a dynamical system \( T = \frac{10}{s+1} \) in a positive feedback loop with \( \Delta = -\frac{1}{20} \). Does the loop satisfy the conditions of the small gain theorem? Provide a visual pictorial. What does this tell us about the stability of the closed loop system? Show that the small gain condition is NOT necessary for closed loop stability by modifying the above given \( \Delta \).

(b) Does there exist a \( \Delta \) with \( |\Delta(j\omega)| = \frac{1}{20} \) which will destabilize the feedback loop? If so, find one. Repeat this for \( |\Delta(j\omega)| = 3 \).

(c) Consider the uncertain MIMO dynamical systems

\[
\begin{align*}
P &= P_o + W_1 \Delta W_2 \\
P &= P_o [I - W_1 \Delta W_2]^{-1} \\
P &= [I - P_o W_1 \Delta W_2]^{-1} P_o
\end{align*}
\]

where \( P_o \) represents a nominal MIMO LTI plant model, \( \Delta \) represents a stable LTI uncertainty satisfying \( \sigma_{\max} |\Delta(j\omega)| \leq 1 \) for all \( \omega \), and \( (W_1, W_2) \) represent stable LTI weighting functions. Consider a negative feedback system with controller \( K \). For each of the above uncertainty descriptions, determine sufficient conditions on \( P_o \) and \( K \) such that the feedback loop is stable for all \( \Delta \) satisfying \( \sigma_{\max} |\Delta(j\omega)| \leq 1 \) for all \( \omega \). Are the conditions necessary? Explain clearly the implication if one of the conditions is violated.

(8) Robust Performance: Additive Modeling Error. Consider the SISO uncertain dynamical system \( P = P_o + W_2 \Delta_2 \) where \( P_o \) represents a nominal SISO plant model and \( \Delta_2 \) represents a stable LTI uncertainty satisfying \( |\Delta_2(j\omega)| \leq 1 \) for all \( \omega \). Note that \( P \) defines a family of plants. Let \( W_1 \) denote a stable LTI weighting function. Let \( S \overset{\text{def}}{=} \frac{1}{1+P_o} \) denote the closed loop sensitivity transfer function. Provide necessary and sufficient conditions on \( P_o \) and a nominal controller \( K \) such that the feedback system with \( P \) and \( K \) satisfies the robust performance condition: \( |W_1 S| < 1 \) for all admissible plants. Explain what can happen if the condition is not satisfied. The above robust performance condition is equivalent to what robust stability condition? (Give a pictorial.) Briefly discuss the structured singular value (ssv) and how it relates to the above.
(9) Bode Integral, Push-Pop Phenomenon, Stability Margins. (a) State the Bode integral formula. What are the underlying assumptions? Are they restrictive or are they generally satisfied. Explain? Explain the formula’s significance. Suppose that the underlying assumptions are satisfied. Let $K$ denote an LTI controller such that $|S| = l_1 < 1$ for $\omega \in [0, \omega_1]$, $|S| = l_2 > 1$ for $\omega \in [\omega_1, \omega_2]$, $|S| = 1$ for $\omega \in [\omega_2, \infty)$. Suppose that the plant has a right half plane pole at $s = 10 + j2$. Explain precisely what happens to $l_2$ if $l_1$ is reduced by a factor of 10.

(b) Give at least three good reasons why we would like the sensitivity transfer function matrix to be small at low frequencies.

(c) Let $S$ and $T$ denote the sensitivity and complementary sensitivity transfer function matrices of a stable unity feedback loop. Suppose that $\|S\|_{H^\infty} = \alpha$ and $\|T\|_{H^\infty} = \beta$. What are the associated gain and phase margin bounds? Illustrate via simple Nyquist plot that the bounds are NOT tight and can be arbitrarily violated.

(10) $H^\infty$ and Generalized Block Diagram. (a) Consider a stable LTI system $G$. How is the $H^\infty$ norm of $G$ defined? Give an equivalent frequency domain expression. Provide two interpretations for the latter expression. It can be shown that the $H^\infty$ norm satisfies the so-called sub-multiplicative property $\|G_1 G_2\|_{H^\infty} \leq \|G_1\|_{H^\infty} \|G_2\|_{H^\infty}$. Explain why this property is important. Which of the following are and are not $H^\infty$ systems: (i) $G = \frac{s^2}{s+1}$, (ii) $G = \frac{s}{s+1} e^{-s\Delta}$, (iii) $G = \frac{3s^3}{s+3}$, (iv) $G = \frac{1}{s}$, (v) $G = e^{-s\Delta}$.

(b) Describe the generalized block diagram. State the general $H^\infty$ design problem. Describe the mixed-sensitivity $H^\infty$ problem. What is the associated generalized plant transfer function matrix? Explain how the weighting transfer function matrices are generally selected? Does there always exist a controller $K$ such that the closed loop map $T_{wz}(P, K)$ satisfies $\|T_{wz}(P, K)\|_{H^\infty} < \gamma$ for any $\gamma > 0$? Explain.

(11) LQR Properties.

- (a) Can $g = \frac{5}{s+1}$ be an LQ loop? Explain.

- (b) Give a simple example which illustrates that $\lim_{\rho \to 0^+} K_\rho = 0$ and $\lim_{\rho \to 0^+} J_{\text{min}} = 0$ if and only if $G_{OL}$ is minimum phase.

- (c) Does $\lim_{\rho \to 0^+} K_\rho = 0$ imply that $\lim_{\rho \to 0^+} G_\rho = 0$ necessarily?

- (d) Suppose that $\lim_{\rho \to 0^+} K_\rho = 0$. Show that there exists an orthonormal matrix $W$ such that $\lim_{\rho \to 0^+} \sqrt{\rho} G_\rho = WM$. A sufficient condition for this is that $G_{OL}$ is minimum phase. Is this condition necessary? Clearly explain.

- (e) Give a simple example which illustrates that $\lim_{\rho \to \infty} G_\rho = 0$ if $A$ is stable.

- (f) How large can an LQ-loop sensitivity, complementary sensitivity, and downward gain margin get? How small can an LQ-loop upward gain margin and phase margin get?

(12) LQG, LTR and Loop Shaping. State the LQG problem and its solution. What closed loop properties do LQG controllers offer? Do LQG controllers generally enjoy the stability robustness properties associated with LQR and KF loops? Describe the LQG/LTRI procedure. Give a proof for LTRI. Give sufficient conditions for LTRI. Are the conditions necessary? (b) Discuss how the LQ loop can be shaped. Specifically, discuss how to match the singular values at (i) low frequencies, (ii) high frequencies, and (iii) all frequencies (Explain).