Problem 1 (Performance and Stability Robustness)

Consider a SISO negative feedback loop with open loop transfer function \( L, S = \frac{1}{1+sT} \), \( T = 1 - S \).

(a) Given upper bounds on the sensitivity and complementary sensitivity (\( |S| < \alpha \), \( |T| < \beta \)), give bounds on upward gain margin \( \Upsilon \text{GM} \), downward gain margin \( \Upsilon \text{GM} \), phase margin \( \text{PM} \). When are the bounds tight?

(b) Given \( \Upsilon \text{GM}, \Upsilon \text{LM}, \) and \( \text{PM} \), give lower bounds for \( ||S||_{\text{H}_{\infty}} \) and \( ||T||_{\text{H}_{\infty}} \).

(c) Suppose the open loop transfer function is non-minimum phase, what can be said about \( ||S||_{\text{H}_{\infty}} \)? If \( L \) possesses a time delay, what can be said about \( ||S||_{\text{H}_{\infty}} \)? If \( L \) is unstable, what can be said about \( ||T||_{\text{H}_{\infty}} \)? Explain.

Now let \( z, p > 0 \) denote an open loop right half plane zero and pole, respectively.

(e) Consider a sensitivity weighting \( W = \left[ \frac{s+\omega_s M_s}{M_s(s+\omega_s)} \right] \left[ \frac{M_s s+\omega_p}{s+\omega_p} \right] \) with \( \omega_s > 0, M_s \geq 1, \epsilon \geq 0, \) and \( \omega_p = f z > M_s^2 \omega_s \). Relate \( \omega_s \) to \( z \) for \( \epsilon = 0 \), \( M_s = 2 \), and \( f \). What can be said for \( f = \frac{1}{2}, 1, 2, \infty \)?

(f) Consider a complementary sensitivity weighting \( W = \left[ \frac{s+\omega_l M_l}{\epsilon s+\omega_l} \right] \left[ \frac{s+\omega_p M_l}{s+\omega_p} \right] \) with \( \omega_l > 0, M_l \geq 1, \epsilon \geq 0, \) and \( \omega_p = f p < \omega_l M_l^2 \). Relate \( \omega_l \) to \( p \) for \( \epsilon = 0 \), \( M_l = 2 \), and \( f \). What can be said for \( f = 0, \frac{1}{2}, 1, 2 \)?

(g) Give necessary and sufficient conditions on \( |L| \) and \( \angle L \) so that \( |S| < \alpha \) (for all \( \omega \)). Now lets fix \( \omega \).

- Give conditions on \( |L| \) such that \( |S| < \alpha \) independent of \( \angle L \). Explain your result with a clear picture?
- Now suppose that the above magnitude conditions on \( |L| \) are not satisfied. Give conditions on \( \angle L \) such that \( |S| < \alpha \) for any \( |L| \)? Explain your result with a clear picture?
- Now suppose that \( \angle L \) does not satisfy the angle condition just obtained. Give conditions on \( |L| \) (that depend on \( \angle L \)) for \( |S| < \alpha \). Explain your result with a clear picture?

Problem 2 (LQR and LQ Servo)

(a) Consider \( P = \left[ \begin{array}{c} b \\ \frac{a}{s-a} \end{array} \right] \). Show how to design an LQ servo that will guarantee zero steady state error to step reference commands. Clearly explain the process. Clearly draw your final LQ servo architecture. Please specify symbolic \( G_p, K_p, J_{\text{min}} \). Hint: \( J = \int_0^\infty (y^2 + pu^2) \, dt, y = M x, M = \left[ \begin{array}{cc} m_1 & m_2 \end{array} \right] \).

(b) For \( a = 1 \) and \( b = 5 \), determine \( \rho \) such that the LQ loop unity gain crossover is \( \omega_g = 3 \text{ rad/sec} \). Determine \( G_L \), upward gain margin \( \Upsilon \text{GM} \), downward gain margin \( \Upsilon \text{GM} \), \( \text{PM} \), and \( \text{DM} \).

(c) What would you expect to happen to the design’s properties as \( \rho \to \infty \)? \( \rho \to 0 \)? Explain. Discuss \( K_p, G_p, J_{\text{min}} \), closed loop poles. Sketch a supporting root square locus. EXTRA CREDIT: \( \omega_g, \text{PM} \)

(d) Can \( G = \frac{1.5(s+0.5)}{p^2} \) be an LQ servo? Explain. What is \( \Upsilon \text{GM} \), \( \Upsilon \text{GM} \), \( \text{PM} \), \( (A, B, M, \rho) \)?

(e) If a PID controller was desired in the final architecture, how would you modify the design process? Briefly discuss using a block diagram. Hint: \( G_{OL} = \left[ \frac{b}{s-a} \right] \left[ \frac{s^2+b_1 s+b_2}{s(s+p)} \right] \)

(f) If \( P = \left[ \begin{array}{c} z, a \end{array} \right] \) with \( z, a > 0 \), and a PI controller was desired in the final architecture, how would you modify the design process? Briefly discuss using a block diagram.

Problem 3 (Kalman Filter)

(a) Consider \( P = \frac{z-a}{z^2 a^2} \). Show how to design a Kalman filter for this system such that the filter tracks steps. Clearly explain the process. Hint: Use LQR solution for dual system. Please specify symbolic \( H_\mu, \Sigma_\mu, J_{\text{min}} = \min E(||x - \hat{x}||^2) \).

(b) For \( a = 1 \) and \( z = 10 \), determine \( \mu \) such that theKF unity gain crossover is \( \omega_g = 3 \text{ rad/sec} \). Determine \( G_{KF} \), upward gain margin \( \Upsilon \text{GM} \), downward gain margin \( \Upsilon \text{GM} \), \( \text{PM} \), and \( \text{DM} \).

(c) What would you expect to happen to the design’s properties as \( \mu \to \infty \)? \( \mu \to 0 \)? Explain. Discuss \( \Sigma_\mu, H_\mu, J_{\text{min}} \), closed loop poles, unity gain crossover \( \omega_g \), \( \text{PM} \). Sketch a supporting root square locus.

(d) Can \( G = \frac{1.5(s+0.5)}{s^2} \) be a KF loop? Explain. What is \( \Upsilon \text{GM} \), \( \Upsilon \text{GM} \), \( \text{PM} \), \( (A, L, C, \mu) \)?