(1) **Feedback System SSR.** Consider a classical negative feedback loop with plant \( P = [A_p, B_p, C_p] \) and controller \( K = [A_k, B_k, C_k] \). Determine a state space representation for the map from the exogenous signals \((r, d_i)\) to the internal signals \((y, u)\).

(2) **Dynamic Augmentation.** Consider a two-input four-output plant \( P = [A_p, B_p, C_p, D_p] \) with input \( u_p \), state \( x_p \), and output \( y_p \). Suppose that the above system is augmented with \( \frac{1}{s} \) (input: \( u_1 \); state: \( x_1 \)) in the first input channel and \((\frac{1}{s+1})^2\) (states: \( x_2, x_3 \); output: \( y_2 \)) in the second output channel. Determine a state space representation for the augmented two-input four-output system with \( u \) denoting the new input, \( x = [x_p^T \ x_1 \ x_2 \ x_3]^T \) the state, and \( y \) the output of the augmented system.

(3) **Coordinate Transformation: Change of Units.** Consider a MIMO plant \( P = [A_p, B_p, C_p, D_p] \) with input \( u_1 \), internal state \( x_1 \), and output \( y_1 \). Suppose that we wish to change the units of \( u_1, x_1, \) and \( y_1 \). Show how this can be done. Let \( u_2, x_2, \) and \( y_2 \) denote the new input, state, and output vectors, respectively. Discuss what fundamental dynamical system properties are invariant under the above transformation.

(4) **Stabilizability, State Feedback, Pole-Placement.** Consider the dynamical system
\[
\dot{x}_1 = px_1 - (1 + p)x_2 + ku \quad \dot{x}_2 = -x_2 + ku \quad y = gx_1.
\]
Is the above system stabilizable? Explain. Discuss what pole-placement attributes can be achieved via constant gain full state feedback; i.e. where can the closed loop poles be placed.

(5) **Model Based Compensator.** Consider the plant \( P = \frac{1-s}{10-s} = [A_p, B_p, C_p, D_p] \) with \( A_p = 10, B_p = 1, C_p = 9, D_p = 1 \). Design a model based compensator which guarantees closed loop stability, zero steady state error to step input disturbances, and achieves dominant closed loop poles at \( s = -1 \pm j1 \) and \( s = -10 \pm j10 \). Hint: Order the states of your augmented system (i.e. design plant) as \( x = [x_p^T \ x_1]^T \). NOTE: You only need to set up the equations. Explain the separation principle.

(6) **Linear Matrix Transformation: SVD.** Consider the non-singular linear matrix transformation \( y = Mx \) where \( M \) is a \( 2 \times 2 \) real matrix. Provide a carefully labelled pictorial which describes how the matrix \( M \) maps vectors \( x \) into vectors \( y \). Use gain concepts to discuss what is meant by certain input directions being maximally amplified and others being minimally amplified.

(7) **SVD for a Dynamical System.** Consider the map \( T_{d,y} \) from torque disturbances \( d_i \) to joint angles \( y \) for a PUMA robotic manipulator. Suppose that a singular value decomposition at 7 rad/sec results in the following directionality information: \( v_1 = [0.1873 \ 0.9823 e^{j79.96^\circ}]^T \), \( \sigma_1 = ||T_{d,y}||_{H^\infty} = 8.01 \), \( u_1 = [0.1673 e^{-j38^\circ} \ 0.9859 e^{j150.94^\circ}]^T \). What does this information tell us about the worst case sinusoidal input disturbance? In what sense is the worst case input disturbance amplified? How is the energy of the output \( ||y||_{L^2} \) related to the energy of the input \( ||u||_{L^2} \)? Describe how you might use \( H^\infty \) optimal control theory to directly improve the above disturbance response. What would you weight? How might you select the weighting function?

(8) **Small Gain Theorem.** Consider a dynamical system \( T = \frac{10}{s+1} \) in a positive feedback loop with \( \Delta = -\frac{1}{s} \). Does the loop satisfy the conditions of the small gain theorem? Provide a
visual pictorial. Is the small gain condition necessary for stability? Is the closed loop system stable? Provide a Δ with \(|Δ(jω)| = \frac{1}{5}\) for all \(ω\) which destabilizes the feedback loop.

(9) Divisive Uncertainty: Robust Stability. Consider the uncertain dynamical system \(P = P_o[1 - WΔ]^{-1}\) where \(P_o\) represents a nominal plant model, \(Δ\) represents LTI uncertainty and \(W\) is an LTI weighting function. Consider a negative feedback system with controller \(K\). Determine sufficient conditions on \(P_o\) and \(K\) such that the above feedback loop is stable for all \(Δ\) satisfying \(σ_{max}[Δ(jω)] \leq 1\) for all \(ω\). Is the condition necessary? Explain clearly.

(10) Robust Performance: Multiplicative Modeling Error. Consider the SISO uncertain dynamical system \(P = P_o(1 + Δ)\) where \(P_o\) represents a nominal plant model and \(Δ\) represents LTI uncertainty satisfying \(σ_{max}[Δ(jω)] \leq 1\) for all \(ω\). Note that \(P\) defines a family of plants. Let \(W\) denote a stable LTI weighting function. Provide necessary and sufficient conditions on \(P_o\) and a nominal controller \(K\) such that the feedback system with \(P\) and \(K\) satisfies the robust performance condition: \(|WS| < 1\) for all admissible plants. Explain what can happen if the condition is not satisfied. Briefly discuss the structured singular value (ssv) and how it relates to the above.

(11) \(H^∞\) Norm. Consider a stable LTI system \(G\). How is the \(H^∞\) norm of \(G\) defined? Give an equivalent frequency domain expression. Provide two interpretations for the latter expression. It can be shown that the \(H^∞\) norm satisfies the so-called sub-multiplicative property \(\|G_1G_2\|_{\H{∞}} \leq \|G_1\|_{\H{∞}}\|G_2\|_{\H{∞}}\). Explain why this property is important. Which of the following are and are not \(H^∞\) systems: (a) \(G = \frac{s^2 + Δ}{s + 1}\), (b) \(G = \frac{s^2}{s + 1}\), (c) \(G = \frac{s^3}{s + 1}\).

(12) Generalized Block Diagram: \(H^∞\) Control Design. Describe the generalized block diagram. State the general \(H^∞\) design problem. Describe the mixed-sensitivity \(H^∞\) problem. What is the associated generalized plant transfer function matrix? Explain how the weighting transfer function matrices are generally selected? Does there always exist a controller \(K\) such that the closed loop map \(T_{wz}(P, K)\) satisfies \(\|T_{wz}(P, K)\|_{\H{∞}} < γ\) for any \(γ > 0\)? Explain.

(13) LQR Properties.

- (a) Can \(g = \frac{10}{s + 9}\) be an LQ loop? Explain.
- (b) Give a simple example which illustrates that \(\lim_{ρ→0^+} K_ρ = 0\) and \(\lim_{ρ→0^+} J_{min} = 0\) if and only if \(G_{OL}\) is minimum phase.
- (c) Suppose that \(\lim_{ρ→0^+} K_ρ = 0\). Show that there exists an orthonormal matrix \(W\) such that \(\lim_{ρ→0^+} \sqrt{ρ}G_ρ = W.M\). A sufficient condition for this is that \(G_{OL}\) is minimum phase. Is this condition necessary?
- (d) Give a simple example which illustrates that \(\lim_{ρ→∞} G_ρ = 0\) if \(A\) is stable.

(14) KF Loop Shaping. Discuss how the KF loop can be shaped. Specifically, discuss how to match the singular values at (a) low frequencies, (b) high frequencies, and (c) all frequencies.

(15) LQG/LTR Properties. State the LQG problem and its solution. Do LQG controllers generally enjoy the stability robustness properties associated with LQR and KF loops? Describe the LQG/LTRO procedure. Give a proof for LTRO. Give sufficient conditions for LTRO.

(16) Bode Integral: Push-Pop Phenomenon. State the Bode integral formula. What are the underlying assumptions? Are they restrictive or are they generally satisfied? Explain. Explain the formula’s significance. Suppose that the underlying assumptions are satisfied. Let
$K$ denote an LTI controller such that $|S| = l_1 < 1$ for $\omega \in [0, \omega_1)$, $|S| = l_2 > 1$ for $\omega \in [\omega_1, \omega_2)$, $|S| = 1$ for $\omega \in [\omega_2, \infty)$. Suppose that the plant has a right half plane pole at $s = 10 + j2$. Explain precisely what happens to $l_2$ if $l_1$ is reduced by a factor of 10.

(17) **Sensitivity Reduction.** Give at least three good reasons why we would like the sensitivity transfer function matrix to be small at low frequencies.

(18) **Margins from Sensitivity Bounds.** Suppose that $\sigma_{\text{max}} [S] < \alpha$ and $\sigma_{\text{max}} [T] < \beta$. What are associated the gain and phase margin bounds? Are the bounds tight? That is, can you provide an example which arbitrarily violates (exceeds) one of the bounds? Hint: Provide a Nyquist plot.