Problem 1 (Gaussian Elimination, Fundamental Subspaces, General Solution)
Consider the system of linear algebraic equations:

\[ \mathbf{Ax} = \mathbf{b} \]

with \( \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \).

(a) What constraints must \( \mathbf{b} \) satisfy for a solution to exist? Hint: Perform Gaussian Elimination
(b) Determine the set of all column vectors \( \mathbf{x} \) such that \( \mathbf{Ax} = \mathbf{0} \). This set is called the right null space of \( \mathbf{A} \) and is denoted \( \mathcal{N}(\mathbf{A}) \).
(c) Determine the set of all row vectors \( \mathbf{w}^T \) such that \( \mathbf{w}^T \mathbf{A} = 0 \). This set is called the left null space of \( \mathbf{A} \) and is denoted \( \mathcal{N}(\mathbf{A}^T) \). How is this set related to the constraints found above on \( \mathbf{b} \)?
(d) Determine the set of all row vectors \( \mathbf{w}^T \mathbf{A} \). This set is called the row space of \( \mathbf{A} \) and is often denoted \( \mathcal{R}(\mathbf{A}^T) \).
(e) Parameterize the set of all solutions \( \mathbf{x} \); i.e. determine the general solution and include constraints on \( \mathbf{b} \).

Problem 2 (Least Squares)
Consider the system of linear algebraic equations:

\[ \mathbf{Ax} = \mathbf{b} \]

with \( \mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

(a) Determine \( \mathbf{x} \) which minimizes \( \| \mathbf{b} - \mathbf{Ax} \| \) where \( \| \mathbf{v} \| \overset{\text{def}}{=} \sqrt{\mathbf{v}^T \mathbf{v}} \).
(b) Is your solution \( \mathbf{x} \) unique? Provide geometric justification for your answer.

Problem 3 (Minimum Norm)
Determine a point on the line \( x_2 = 1 - x_1 \) which is closest to the origin. Show that this is equivalent to \( \min_{\mathbf{Ax} = \mathbf{b}} \| \mathbf{x} \| \) for some \( \mathbf{A} \) and \( \mathbf{b} \).

Problem 4 (SVD)
Consider the matrix mapping:

\[ \mathbf{y} = \mathbf{Mx} \]

with \( \mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 10 \end{bmatrix} \), \( \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \), and \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \).

(a) Use MATLAB to show that the unit circle \( x_1^2 + x_2^2 = 1 \) gets mapped onto an ellipse in the \( y \)-plane. Sketch the ellipse.
(b) What is the length of the ellipse’s major axis? minor axis?
(c) Determine a unit vector along the ellipse’s major axis? minor axis?
(d) Determine a unit vector on the unit circle in the \( x \)-plane that gets mapped onto the ellipse’s major axis in the \( y \)-plane.
(e) Determine a unit vector on the unit circle in the \( x \)-plane that gets mapped onto the ellipse’s minor axis in the \( y \)-plane.
(f) Show how MATLAB’s svd command can help you answer (a)-(e) above.

Problem 5 (State Space Realization and Arithmetic)
(a) Sketch a block diagram for for the following LTI system:

\[ H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_3 s^2 + a_1 s + a_0}. \] (1)

Indicate state variables on your diagram. Determine a state space realization. Hint: See controller canonical form within text.
(b) Determine a state space representation for the feedback system with external signals \((r, d_i, d_o, n)\), state \(x = [x_p \; x_k]^T\), and outputs \((e, u, y)\) defined by the following equations:

\[
\begin{align*}
  e &= r - y - n \\
  \dot{x}_k &= A_k x_k + B_k e \\
  u &= C_k x_k \\
  \dot{x}_p &= A_p x_p + B_p u_p \\
  y &= C_p x_p + d_o.
\end{align*}
\]

Problem 6 (Matrix Exponential)
Consider the matrix \(A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\).

(a) Determine all eigenvalues \(s_i\) of \(A\). Hint: \(\det(s_i I - A) = 0, \; i = 1, 2\)

(b) Determine right eigenvectors of \(A\) corresponding to the above eigenvalues. Hint: \((s_i I - A)v_i = 0, \; v_i \neq 0\).

(c) Form the matrix of right eigenvectors \(V = [v_1 \; v_2]\). Determine the matrix \(W = V^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}\). Show that \(w_i^T A = s_i w_i^T\). \(W\) thus contains the left eigenvectors of \(A\).

(d) Compute \(e^{At} = I + At + \frac{A^2 t^2}{2} + \cdots\).

(e) Form \(sI - A\). Compute \((sI - A)^{-1}\). Compute \(e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}\). How does your answer compare with that obtained in (d)?

(f) Show that \(e^{At} = \sum_{i=1}^{2} e^{s_i t} v_i w_i^T\); i.e. this is a third way to compute the matrix exponential. Compare with answers obtained in (d) and (e).

Problem 7 (Modal Analysis)
Consider the LTI system:

\[
\dot{x} = Ax \\
\]

with \(A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\). See notation in Problem 6.

(a) Determine \(x(t) = e^{At} x_o\) when \(x_o = v_1\). Hint: \(w_2^T v_1 = 0\)

(b) Determine \(x(t) = e^{At} x_o\) when \(x_o = v_2\). Hint: \(w_1^T v_2 = 0\)

Problem 8 (State Computation)
Consider the LTI system:

\[
\dot{x} = Ax + Bu
\]

with \(A = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\).

(a) Sketch a block diagram for the system.

(b) Compute \(x(t)\) when \(u = 1(t)\) (unit step function) and \(x(0) = 0\). Hint: Propagate \(u\) through your block diagram.

(c) Form \((sI - A)^{-1}BU(s)\). Determine \(x(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}BU(s)\}\). How does this compare with answer obtained in (b)?

Problem 9 (Transmission Zeros)
Consider the LTI system:

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

with \(A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\).

(a) Sketch a block diagram for the system.

(b) Determine the system transfer function.

(c) Determine the associated transmission zeros.

(d) Use (c) to determine an input \(u(t)\) such that the steady state output is zero; i.e. \(y_{ss} = 0\).

(e) Repeat the above with \(C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\).
**Problem 10 (Linearization)**
Consider the following nonlinear model for a fishery:

\[ \dot{x} = f(x, u) \]

with

\[ f(x, u) = rx(1 - \frac{x}{k}) - qxu. \]

NOTE: Here, \( r > 0 \) represents growth rate, \( k > 0 \) represents the maximum number of fish (referred to as carrying capacity), \( q > 0 \) represents catchability, \( x \in [0, k] \) represents the fish biomass, and \( u \geq 0 \) represents harvesting effort.

(a) Determine all system equilibria.
(b) Linearize the above nonlinear system about the equilibrium associated with \( u_e \in (0, r/q) \) and \( x_e \in (0, k) \).
(c) Discuss the stability of the resulting linear system. Discuss how you would use the linear model to approximate the nonlinear model.
(d) Suppose that \( r = k = q = 1 \). Consider the linear model associated with \( u_e = 0 \). Suppose that the initial condition for the nonlinear model is \( x(0) = x_0 = 0.5 \). Use Simulink to determine the constant value \( \delta u = u - u_e \) when the response using the linear approximation \( \dot{x} = x_e + \delta x \) deviates from that associated with the nonlinear model \( x \) by more than 10% at any time instant.

**Problem 11 (Feedback Compensation)**
Consider the LTI plant

\[ \mathcal{P}(s) = \frac{1}{s - 1} \left[ \frac{p}{s + p} \right] \left[ \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right] \]

with \( p = 100, \zeta = 0.1, \omega_n = 100 \), the LTI controller

\[ K(s) = \frac{3(s + \frac{2}{3})}{s} \left[ \frac{100}{s + 100} \right] \]

and the LTI reference command pre-filter

\[ W(s) = \frac{s + \frac{2}{3}}{s + \frac{2}{3}}. \]

(a) Use MATLAB to determine the closed loop poles as well as the transfer functions \( T_{ry}, T_{dy}, T_{d\phi}, T_{ny} \).
(b) Let

\[ L \overset{\text{def}}{=} PK \]

denote the open loop transfer function. Use MATLAB to sketch Bode plots for \( |L|, \angle L \). Specify your gain and phase stability margins. Hint: See allmargin command.
(c) Let

\[ S \overset{\text{def}}{=} \frac{1}{1 + L} \]

denote the sensitivity transfer function and

\[ T \overset{\text{def}}{=} 1 - S \]

the complementary sensitivity transfer function. Use MATLAB to sketch Bode plots for \( |S|, |T|, |KS|, |PS|, |T_{ry}| \). Interpret each of your plots in terms of low frequency command following and disturbance attenuation as well as high frequency noise attenuation.
(d) Use Simulink to determine the response \( y \) to a step reference command \( r \) for \( p = 100, 10, 5 \).