EEE582 Homework 1-2, Fall 2007 A.A. Rodriguez
Due September 19 GWC 352, 965-3712

Problem 1 (Gaussian Elimination, Fundamental Subspaces, General Solution)
Consider the system of linear algebraic equations:

$$Ax = b$$

with $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

(a) What constraints must b satisfy for a solution to exist? Hint: Perform Gaussian Elimination

(b) Determine the set of all column vectors x such that $Ax = 0$. This set is called the right null space of A and is denoted $\mathcal{N}(A)$.

(c) Determine the set of all row vectors w^T such that $w^T A = 0^T$. This set is called the left null space of A and is denoted $\mathcal{N}(A^T)$. How is this set related to the constraints found above on b?

(d) Determine the set of all row vectors $w^T A$. This set is called the row space of A and is often denoted $\mathcal{R}(A^T)$.

(e) Parameterize the set of all solutions x; i.e. determine the general solution and include constraints on b. ■

Problem 2 (Least Squares)
Consider the system of linear algebraic equations:

$$Ax = b$$

with $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) Determine x which minimizes $\|b - Ax\|$ where $\|v\| \stackrel{\text{def}}{=} \sqrt{v^T v}$.

(b) Is your solution x unique? Provide geometric justification for your answer. ■

Problem 3 (Minimum Norm)
Determine a point on the line $x_2 = 1 - x_1$ which is closest to the origin. Show that this is equivalent to $\min_{Ax=b} \|x\|$ for some A and b. ■

Problem 4 (SVD)
Consider the matrix mapping:

$$y = Mx$$

with $M = \begin{bmatrix} 1 & 1 \\ 0 & 10 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) Use MATLAB to show that the unit circle $x_1^2 + x_2^2 = 1$ gets mapped onto an ellipse in the y-plane. Sketch the ellipse.

(b) What is the length of the ellipse's major axis? minor axis?

(c) Determine a unit vector along the ellipse's major axis? minor axis?

(d) Determine a unit vector on the unit circle in the x-plane that gets mapped onto the ellipse's major axis in the y-plane.

(e) Determine a unit vector on the unit circle in the x-plane that gets mapped onto the ellipse's minor axis in the y-plane.

(f) Show how MATLAB's svd command can help you answer (a)-(e) above. ■

Problem 5 (State Space Realization and Arithmetic)
(a) Sketch a block diagram for for the following LTI system:

$$H(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o}. \tag{1}$$

Indicate state variables on your diagram. Determine a state space realization. Hint: See *controller canonical form* within text.

(b) Determine a state space representation for the feedback system with external signals (r, d_i, d_o, n), state $x = [\ x_p \quad x_k\ ]^T$, and outputs (e, u, y) defined by the following equations:

$$e = r - y - n \qquad u_p = u + d_i \tag{2}$$
$$\dot{x}_k = A_k x_k + B_k e \qquad \dot{x}_p = A_p x_p + B_p u_p \tag{3}$$
$$u = C_k x_k \qquad y = C_p x_p + d_o. \tag{4}$$

■

Problem 6 (Matrix Exponential)
Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Determine all eigenvalues s_i of A. Hint: $\det(s_i I - A) = 0$, $i = 1, 2$

(b) Determine right eigenvectors of A corresponding to the above eigenvalues. Hint: $(s_i I - A)v_i = 0$, $v_i \neq 0$.

(c) Form the matrix of right eigenvectors $V = [\ v_1 \quad v_2\ ]$. Determine the matrix $W = V^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}$. Show that $w_i^T A = s_i w_i^T$. W thus contains the left eigenvectors of A.

(d) Compute $e^{At} = I + At + \frac{A^2 t^2}{2} + \cdots$.

(e) Form $sI - A$. Compute $(sI - A)^{-1}$. Compute $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$. How does your answer compare with that obtained in (d)?

(f) Show that $e^{At} = \sum_{i=1}^{2} e^{s_i t} v_i w_i^T$; i.e. this is a third way to compute the matrix exponential. Compare with answers obtained in (d) and (e). ■

Problem 7 (Modal Analysis)
Consider the LTI system:

$$\dot{x} = Ax \qquad x(0) = x_o$$

with $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. See notation in Problem 6.

(a) Determine $x(t) = e^{At} x_o$ when $x_o = v_1$. Hint: $w_2^T v_1 = 0$

(b) Determine $x(t) = e^{At} x_o$ when $x_o = v_2$. Hint: $w_1^T v_2 = 0$ ■

Problem 8 (State Computation)
Consider the LTI system:

$$\dot{x} = Ax + Bu$$

with $A = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Sketch a block diagram for the system.
(b) Compute $x(t)$ when $u = 1(t)$ (unit step function) and $x(0) = 0$. Hint: Propagate u through your block diagram.
(c) Form $(sI - A)^{-1} BU(s)$. Determine $x(t) = \mathcal{L}^{-1}\{(sI - A)^{-1} BU(s)\}$. How does this compare with answer obtained in (b)? ■

Problem 9 (Transmission Zeros)
Consider the LTI system:

$$\dot{x} = Ax + Bu \qquad y = Cx$$

with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = [\ -1 \quad 1 \quad 0\ ]$.

(a) Sketch a block diagram for the system.
(b) Determine the system transfer function.
(c) Determine the associated transmission zeros.
(d) Use (c) to determine an input $u(t)$ such that the steady state output is zero; i.e. $y_{ss} = 0$.
(e) Repeat the above with $C = [\ 1 \ \ 0 \ \ 1\ ]$. ■

Problem 10 (Linearization)
Consider the following nonlinear model for a fishery:

$$\dot{x} = f(x, u)$$

with $f(x, u) = rx(1 - \frac{x}{k}) - qxu$.

NOTE: Here, $r > 0$ represents growth rate, $k > 0$ represents the maximum number of fish (referred to as carrying capacity), $q > 0$ represents catchability, $x \in [0, k]$ represents the fish biomass, and $u \geq 0$ represents harvesting effort.

(a) Determine all system equilibria.
(b) Linearize the above nonlinear system about the equilibrium associated with $u_e \in (0, \frac{r}{q})$ and $x_e \in (0, k)$.
(c) Discuss the stability of the resulting linear system. Discuss how you would use the linear model to approximate the nonlinear model.

(d) Suppose that $r = k = q = 1$. Consider the linear model associated with $u_e = 0.5$. Suppose that the initial condition for the nonlinear model is $x(0) = x_o = 0.5$. Use Simulink to determine the constant value $\delta u = u - u_e$ when the response using the linear approximation $\hat{x} = x_e + \delta x$ deviates from that associated with the nonlinear model x by more that 10% at any time instant. ■

Problem 11 (Feedback Compensation)
Consider the LTI plant

$$P(s) = \frac{1}{s-1}\left[\frac{p}{s+p}\right]\left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]$$

with $p = 100$, $\zeta = 0.1$, $\omega_n = 100$, the LTI controller

$$K(s) = \frac{3(s + \frac{2}{3})}{s}\left[\frac{100}{s+100}\right]$$

and the LTI reference command pre-filter

$$W(s) = \frac{\frac{2}{3}}{s + \frac{2}{3}}.$$

(a) Use MATLAB to determine the closed loop poles as well as the transfer functions T_{ry}, $T_{d_i y}$, $T_{d_o y}$, T_{ny}.
(b) Let

$$L \stackrel{\text{def}}{=} PK$$

denote the *open loop transfer function*. Use MATLAB to sketch Bode plots for $|L|$, $\angle L$. Specify your gain and phase stability margins. Hint: See allmargin command.
(c) Let

$$S \stackrel{\text{def}}{=} \frac{1}{1+L}$$

denote the *sensitivity transfer function* and

$$T \stackrel{\text{def}}{=} 1 - S$$

the *complementary sensitivity transfer function*. Use MATLAB to sketch Bode plots for $|S|$, $|T|$, $|KS|$, $|PS|$, $|T_{ry}|$. Interpret each of your plots in terms of low frequency command following and disturbance attenuation as well as high frequency noise attenuation.
(d) Use Simulink to determine the response y to a step reference command r for $p = 100, 10, 5$. ■