Problem 1 (Gaussian Elimination, Fundamental Subspaces, General Solution, Projections)
Consider the system of linear algebraic equations:
\[ Ax = b \]
with
\[
A = \begin{bmatrix}
0 & 1 & 1 \\
0 & -1 & -3 \\
0 & -1 & -9
\end{bmatrix}
\]
and
\[
b = \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}.
\]

(a) Parameterize the set of all solutions \( x \).

(b) Determine a basis for each of the four (4) fundamental subspaces: \( \mathcal{R}(A) \), \( \mathcal{R}(A^T) \), \( \mathcal{N}(A) \), \( \mathcal{N}(A^T) \).

(c) Determine a solution that lies within the range space of \( A^T \). What can be said about such a solution?

(d) Given that \( b \) does not lie within the range space of \( A \), determine the set of all \( x \) which minimizes the Euclidean norm (distance) \( \| b - Ax \| \). Amongst these, how would you determine the minimum norm solution?

Problem 2 (State Space Realization and Arithmetic)
(a) Classic Feedback System. Determine a state space representation for the feedback system with external signals \((r, d_i, d_o, n)\), state \( x = [x_p^T \ x_k^T \ x_s^T \ x_w^T]^T \), and outputs \((e, u, y)\) defined by the following equations:
\[
\begin{align*}
e &= \hat{r} - y_s \\
\dot{x}_k &= A_k x_k + B_k e \\
u &= C_k x_k + D_k e \\
\dot{x}_w &= A_w x_w + B_w r \\
\dot{x}_s &= A_s x_s + B_s (y + n)
\end{align*}
\]

with \( D_p = D_k = 0 \). Here, \((x_p, x_k, x_s, x_w)\) represent plant, controller, sensor, and pre-filter states.

(b) Full State Feedback. Consider the following system with full state feedback:
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u \\
y &= x_1 \\
u &= r - g_1 x_1 - g_2 x_2
\end{align*}
\]

Sketch a block diagram. Determine the transfer function \( T_{ry} \). Choose \( g_1 \) and \( g_2 \) such that the closed loop poles are \(-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}\).

(c) Model Based Observer. Consider the following model based observer:
\[
\begin{align*}
\dot{x}_1 &= \hat{x}_2 + h_1 (y - \hat{x}_1) \\
\dot{x}_2 &= h_2 (y - \hat{x}_1) \\
\hat{y} &= \hat{x}_1
\end{align*}
\]

Sketch a block diagram. Determine the transfer function \( T_{\hat{y}y} \). Choose \( h_1 \) and \( h_2 \) such that the closed loop poles are \(-1 \pm j\).

Problem 3 (Modal Analysis)
Consider the linear dynamical system:
\[ \dot{x} = Ax \]
\[ x(0) = x_o \]
with \( A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -25 \\ 0 & 1 & -8 \end{bmatrix} \). (a) Determine the system’s characteristic equation? (b) Determine all eigenvalues. (c) Determine a set of linearly independent eigenvectors. Is \( A \) diagonalizable? (d) Compute a (symbolic) modal expression for \( x \). (e) Determine how each mode can be independently excited via carefully selected (general and real) initial conditions. Determine the response for each initial condition.

**Problem 4 (State Computation)**

Consider the linear system:

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

with \( A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = I_{7 \times 7} \). (a) Sketch a block diagram for the system. (b) Compute \( x \) and \( y \) when \( u = \delta(t) \) (unit impulse) and \( x(0) = 0 \).

**Problem 5 (Transmission Zeros)**

Consider the linear time invariant (LTI) system:

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

with \( A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 1 \\ 1 & 2 \end{bmatrix} \). (a) Sketch a carefully labeled block diagram for the system. (b) Determine the system transfer function matrix. (c) Determine the associated system poles. (d) Determine the associated system transmission zeros and the associated input and state directions. (e) Use the above to compute the associated (general and real) \( x_o, u(t), x(t), \) and \( y(t) \).

**Problem 6 (Controllability)**

Consider the LTI system defined by the state space representation:

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

with \( A = \begin{bmatrix} -1 & 0 & 0 \\ (a-1)(b-2) & -2 & 0 \\ a-1 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ b-2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} a-1 & 1 & c-3 \end{bmatrix} \) where the parameters \( a, b, c \) are specified below.

(a) Sketch a block diagram for the system. Identify the associated subsystems and their transfer functions. (b) Use result in (a) (and nothing else!) to determine the values of \( a, b, c \) for which the system uncontrollable.

Now suppose \( a = 1, b = c = 4 \).

(c) Compute the system eigenvalues and a set of mutually orthonormal right and left eigenvectors.

(d) Is the system controllable? Explain using multiple tests; i.e. PZCs, matrix rank test, PBH modal test.

(c) Determine the set of all states that are reachable from \( x(0) = 0 \)?

Is \( x(1) = [0 \ 1 \ 1]^T \) reachable from \( x(0) = 0 \)? Explain.

If not, determine the closest reachable state. Determine the associated minimum energy control to achieve the relevant transfer. Can another control law achieve the relevant transfer? Explain.

Is \( x(1) = [0 \ 0 \ 1]^T \) reachable from \( x(0) = 0 \)? Explain.
If not, determine the closest reachable state. Determine the associated minimum energy control to achieve the relevant transfer. Can another control law achieve the relevant transfer? Explain. How would you achieve the relevant transfer in practice?

(d) What closed loop poles can be achieved with full state feedback? Explain. What is the fastest achievable settling time?

Now suppose $a = 2$, $b = c = 4$. Repeat (a)-(d).

**Problem 7 (Observability)**
Consider the system in Problem 6. For $a = 1$ and $a = 2$ (with $b = c = 4$) answer each of the following.
(a) Compute system eigenvalues and a set of mutually orthonormal right-left eigenvectors.
(b) Is the system observable? Explain your answer using multiple tests.
(c) Determine a (symbolic) modal expression for $y$ when $u = 0$ and $x(0) = x_0$.
(d) Using the expression derived above, can $x(0) = x_0$ be determined uniquely from $y$?
If yes, show how to determine $x_0$; e.g. via Grammian.
If not, show how to determine the set of all possible $x_0$; e.g. via observability matrix and modal concepts.
(e) Discuss how a model based observer matrix $H$ can be used to place the eigenvalues of $A - HC$. Clearly point out any limitations that may exist. What can be said about the fastest achievable settling time for the associated state estimation error dynamics $\tilde{x} = (A - HC)\tilde{x}$ where $\tilde{x} = x - \bar{x}$?

**Problem 8 (Model Based Compensator)**
Consider the linear time invariant (LTI) plant

$$P(s) = \frac{10 - s}{s^2 - s + 1} = [A_p, B_p, C_p, D_p]$$

with $(A_p, B_p, C_p, D_p), (u_p, x_p, y_p)$, in controller canonical form.
Show how to design a model based compensator which satisfies the following design specifications:
(i) zero steady state error to ramp reference commands,
(ii) zero steady state error to sinusoidal input disturbances $A\cos(\omega_0 t + \theta)$,
(iii) dominant closed loop poles at $s = -4 \pm j3, -5$,
(iv) the remainder of the closed loop poles near $s = -30$ or $s = -30 \pm j30$
Clearly describe the design process and the form of the final controller. Hint: Don’t compute $G$ and $H$!

**Problem 9 (SVD)**
Consider the SVD $M = USV^H$ where:

$$U = \begin{bmatrix}
-0.7063 & 0.2934 & 0.6443 \\
-0.4604 & 0.5009 & -0.7328 \\
-0.5377 & -0.8143 & -0.2187 \\
\end{bmatrix}$$

$$S = \begin{bmatrix}
2.5353 & 0 & 0 & 0 \\
0 & 0.9910 & 0 & 0 \\
0 & 0 & 0.5638 & 0 \\
\end{bmatrix}$$

$$V = \begin{bmatrix}
-0.4184 & 0.5947 & -0.2956 & 0.6190 & 0.0279 \\
-0.3900 & 0.5099 & 0.1840 & -0.6492 & -0.3642 \\
-0.3800 & -0.4278 & -0.7641 & -0.2009 & -0.2200 \\
-0.5033 & -0.4322 & 0.5212 & 0.3426 & -0.4134 \\
-0.5245 & -0.1288 & 0.1524 & -0.1942 & 0.8045 \\
\end{bmatrix}$$

(a) Discuss the maximum and minimum input-output amplification directionality properties of $M$.
(b) Determine a basis for each of the four fundamental subspaces of $M$. 
