Problem

Reduced Order Observers

Reduced Order Model Based Compensators

Reference: Introduction to Dynamical Systems: Theory, Models & Applications
David G. Luenberger
John Wiley & Sons 1979
pp 304

Consider the dynamical system:
\[
\begin{align*}
\dot{z} &= \hat{A}z + \hat{B}u \\
y &= \hat{C}z
\end{align*}
\]

Let \( x = Qz \).

a) Compute \( A, B, C \) in terms of \( \hat{A}, \hat{B}, \hat{C}, \hat{Q} \).

b) Suppose \( \text{rank} \hat{C} = p \); i.e. full row rank.
   How would you construct a nonsingular \( Q \) so that \( x = \begin{bmatrix} w \\ y \end{bmatrix} \).

Hint: Choose \( Q = \begin{bmatrix} M \\ \hat{C} \end{bmatrix} \) so \( w = Mz \).

Let the resulting \( A \times B \) be partitioned as follows:

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
\]

c) What does \( C \) look like?

d) Show that \( (\hat{A}, \hat{C}) \) is detectable \( \iff (A_{11}, A_{21}) \) is detectable.

Now let's motivate the observer problem.

We have access to \( y \) (part of the state vector \( x \)).
We do not have access to \( w \).
How can we obtain an estimate for \( w \). Call the estimate \( \hat{w} \).
Luenberger's Idea (In his PhD thesis)

- Try to estimate \( v = w - H y \) where \( H \in \mathbb{C}^{r \times p} \)
is a design parameter.

Call the estimate \( \hat{v} \).

Then take \( \hat{w} = \hat{v} + H y \)

What made him think of this? Perhaps he noted that
\[
    w - \hat{w} = [v + H y] - [\hat{v} + H y] \\
    = v - \hat{v}
\]

Consequently, \( \hat{w} \) will be close to \( w \) if and only if \( \hat{v} \) is close to \( v \).

e) Write down a differential equation for \( v \) in terms of \( y \) and \( u \);
    i.e., determine the matrices \( M_1, M_2, M_3 \).
    where
    \[
    \dot{v} = M_1 v + M_2 y + M_3 u \\
    w = v + Hy
    \]

The objective of the observer is to take the given signals
\( u + y \) and generating the state estimate: \( \hat{x} = [\hat{w}] \)

Visualization:

\[
\begin{array}{ccc}
\text{u} & \rightarrow & \text{Reduced} \\
& \rightarrow & \text{Order} \\
& \rightarrow & \text{Observer} \\
y & \rightarrow & \hat{w}
\end{array}
\]

The equations for Luenberger's Reduced Order Observer are as follows:
\[
    \dot{\hat{v}} = M_1 \hat{v} + M_2 y + M_3 u \\
    \hat{w} = \hat{v} + H y
\]
It is called a reduced order observer since it is an \((n-p)\)th order system in contrast to the standard \(n\)th order observer.

f) Show that the reduced order observer may be visualized as follows:

\[ \text{Specify } N_1, N_2, \ldots, N_7, N_8 \text{ in terms of the } A_{ij}, B_i, \text{ and } H. \]

This visualization is useful because it makes it clear that like the standard full-order observer which we have studied, Luenberger's reduced order observer is also a feedback system.
9) Show that
\[
\frac{d}{dt} [\hat{w} - \hat{\hat{w}}] = [A_{11} - H A_{21}] [w - \hat{w}]
\]

10) Give necessary and sufficient conditions on the pair \((A_{11}, A_{21})\) so that we can guarantee the existence of a matrix \(H \in \mathbb{C}^{(n-p) \times p}\) such that \(\lim_{t \to \infty} [w(t) - \hat{w}(t)] = 0\).

If the condition is satisfied, what does this imply about the original realization \((\hat{A}, \hat{B}, \hat{C})\)?

Now suppose we want to integrate our reduced order observer ideas with our full-state feedback ideas.

The following block diagram suggests how this should be done:

Here \(G\) represents our state feedback gain matrix.

From the above figure, it follows naturally to call the system within \([\cdot]\) our reduced order model based compensator.
This then gives us

\[ r \rightarrow e \rightarrow K_{HBC}^{red}(s) \rightarrow u \rightarrow P(s) \]

where \( K_{HBC}(s) \) is given by

\[ e \rightarrow -I \rightarrow -e \rightarrow \text{Reduced Order Observer} \rightarrow \hat{w} \rightarrow -G \rightarrow u \]

i) If one lets \( G = [G_1, G_2] \), then the equations for the reduced order model based compensator become

\[
\begin{align*}
\dot{\hat{w}} &= M_1 \hat{w} - M_2 e + M_3 u \\
\dot{\hat{e}} &= \hat{e} - H e
\end{align*}
\]

\[ u = -G_1 \hat{w} + G_2 e \]

Suppose that \( x_R = \hat{w} \) and

\[
\begin{align*}
\dot{x}_R &= A_R x_R + B_R e \\
u &= C_R x_R + D e
\end{align*}
\]

Determine \( A_R, B_R, C_R, D \) in terms of \( A_i, B_i, H, \) and \( G_i \).
3) Show that the closed loop system poles are precisely the eigenvalues of \( A-BG \) and those of \( A_{11}-H_{A_{21}} \).

Hint:
Find these

\[
\frac{dx}{dt} = \begin{bmatrix} x \\ v-\hat{v} \end{bmatrix} = \begin{bmatrix} \phantom{+} & \phantom{+} \\ + & \phantom{+} \end{bmatrix} \begin{bmatrix} x \\ v-\hat{v} \end{bmatrix} + \begin{bmatrix} \phantom{+} & \phantom{+} \\ + & \phantom{+} \end{bmatrix}
\]

Remember that \( x = \begin{bmatrix} w \\ y \end{bmatrix} \)

\[
\dot{x} = Ax + Bu \quad u = -G_1\hat{w} + G_2e \quad e = r - y \quad \Rightarrow \frac{dx}{dt} = \ldots
\]

\[
\dot{v} = H_1v + H_2y + H_3u \quad \left\{ \begin{array}{l}
e = r - y \quad \Rightarrow \frac{dv}{dt} (v-\hat{v}) = \ldots
\end{array} \right.
\]
Problem Minimality

Consider the LTI system

\[ G(s) = [A, B, C, D]. \quad \text{(i.e. } G(s) = C(sI-A)^{-1}B + D) \]

where \( A \in \mathbb{C}^{n \times n}, \ B \in \mathbb{C}^{n \times m}, \ C \in \mathbb{C}^{p \times n}, \ D \in \mathbb{C}^{p \times m}. \)

Dfn:

\([A, B, C, D]\) is said to be a minimal realization (or a minimum order realization) for \( G(s) \) if given any other realization \([\hat{A}, \hat{B}, \hat{C}, \hat{D}]\) for \( G(s) \),

i.e.

\[ G(s) = \hat{C}(sI-\hat{A})^{-1}\hat{B} + \hat{D} \]

where \( \hat{A} \in \mathbb{C}^{n_1 \times n_1}, \ \hat{B} \in \mathbb{C}^{n_1 \times m}, \ \hat{C} \in \mathbb{C}^{p \times n_1}, \ \hat{D} \in \mathbb{C}^{p \times m} \),

then \( n_1 > n \).

If \( n_1 < n \) for some realization \([\hat{A}, \hat{B}, \hat{C}, \hat{D}]\) then the realization \([A, B, C, D]\) is said to be non-minimal.

a) Show that

\((A;B,C,D)\) is a minimal realization \(\iff\) \((A;B)\) is controllable and \((A;C)\) is observable.

b) Indicate which direction is easy \(\Rightarrow\) or \(\Leftarrow\)? Prove easy direction first.

c) Which direction is "hard" \(\Rightarrow\) or \(\Leftarrow\)?

Now prove the "hard" direction. Make sure you explain everything!
b) Suppose that $[A\ B\ C\ D]$ is a minimal realization for the transfer function matrix (tfm) $G(s)$. Show that

$[\hat{A}\ \hat{B}\ \hat{C}\ \hat{D}]$ is also a minimal realization for $G(s)$ if and only if there exists a nonsingular $Q$ such that:

$\hat{A} = QAQ^{-1}$
$\hat{B} = QB$
$\hat{C} = CQ^{-1}$
$\hat{D} = D$

i.e. All minimal realizations for a given tfm are similar.