Problem 1 (Linear Algebra)

Consider the system of linear algebraic equations: \( Ax = b \) with \( A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{bmatrix} \) and \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \).

(a) Parameterize the set of all solutions \( x \).

(b) Determine a basis for each of the four (4) fundamental subspaces: \( \mathcal{R}(A) \), \( \mathcal{R}(A^T) \), \( \mathcal{N}(A) \), \( \mathcal{N}(A^T) \).

(c) Given that \( b \) does not lie within \( \mathcal{R}(A) \), show how to determine the set of all \( x \) which minimizes the Euclidean norm (distance) \( \| b - Ax \| \). Amongst these, show how would you determine the minimum norm solution? Clearly discuss what is unique and what is not!

(d) Modal Analysis. Now consider \( \dot{x} = Ax \), \( x(0) = x_o \). Determine \( x(t) \). Suppose that \( A \) has an eigenvalue \( \sigma + j\omega \) and associated eigenvector \( v = \frac{1}{\sqrt{3}}[e^{-30^o}e^{j120^o} - j]^T \). Show how to excite the mode by specifying a general real initial condition \( x(0) \) and the associated \( x(t) \). Compute \( x(0) \) and \( x(t) \).

(e) SVD. Now consider \( M = U \Sigma V^H \) where \( U = \begin{bmatrix} -0.5 & 0.866 \\ 0.866 & 0.5 \end{bmatrix} \), \( \Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 0.1 \end{bmatrix} \), \( V = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix} \).

Use the above SVD to clearly interpret the mapping \( y = Mx \). Hint: Provide a figure and key relations.

Problem 2 (Controllability, Observability, MBC)

Consider the LTI system \( A_p = \begin{bmatrix} -2 & -8 & 1 \\ 0 & -10 & 1 \\ 0 & 0 & 0 \end{bmatrix} \), \( B_p = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \), \( C_p = \begin{bmatrix} 8 & -8 & 1 \end{bmatrix} \), \( D_p = 0 \).

(a) Sketch a clear block diagram for the system. What is the system transfer function. Discuss pole-zero cancelations and their implications.

(b) Is the system controllable? Explain your answer using multiple tests. Which modes are controllable? uncontrollable? Explain!

(c) Determine a symbolic expression for \( x(t) \) when \( x(0) = 0 \). Determine the set of all states that are reachable from \( x(0) = 0 \)?

(d) Suppose \( x_f = x(1) \) is not reachable from \( x(0) = 0 \). Show how to determine the closest reachable state \( \hat{x}_f \). Now show how to construct a minimum energy control law to reach \( \hat{x}_f \).

(e) Specify the precise form of \( \det(sI - A_p + B_pG) \). Hint: Need not compute determinant!

(f) Is the system observable? Explain your answer using multiple tests. Which modes are observable? unobservable? Explain!

(g) Determine a symbolic expression for \( y(t) \) when \( u(t) = 0 \) and \( x(0) = x_o \).

(h) Discuss two methods to determine \( x_o \). Can \( x_o \) be determined uniquely?

State Estimator

(i) Specify the form of \( \det(sI - A_p + HC_p) \). Hint: Need not compute determinant.

(j) Show how to design a state estimator with settling time 5 seconds. Hint: Show state estimator equations and show how to compute filter gain matrix \( H \). You need not compute \( H \)!

Invariance Concepts

(k) Can \( G \) be selected such that \( (A_p - B_pG, C_p) \) is observable? If so, show how to determine such a \( G \). You need not compute it! If not, explain! Can \( H \) be selected such that \( (A_p - HC_p, B_p) \) is controllable? If so, show how to determine such a \( H \). If not, explain! You need not compute it!

Canonical Forms

(l) If the above system transfer function is implemented in controller or controllability canonical form, what can be said about the resulting state space representation? Hint: You need not give a state space representation!

(m) If the above system transfer function is implemented in observer or observability canonical form, what can be said about the resulting state space representation? Hint: You need not give a state space representation!

Model Based Compensator

(n) We wish to design a model based compensator (MBC) for the above system such that the closed loop
system exhibits zero steady state error to ramp reference commands \( r \), ramp output disturbances \( d_o \) and step input disturbances \( d_i \). We desire a settling time \( t_s \approx 5 \text{ sec} \) and a step overshoot near 4.321\%. Can this be accomplished? If so, explain why and show the design process. Hint: Show how to compute \( G \) and \( H \). Don’t compute them! Specify the design plant \( P_d = [A, B, C, D] \), two relevant characteristic equations (specify all roots), final controller \( K \) and show how to determine the controller zeros.

Problem 3 (Transmission Zeros)
Consider the LTI system \( \dot{x} = Ax + Bu, y = Cx + Du, \) \( A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, B = I_{2 \times 2}, C = \begin{bmatrix} -1 & 10 \\ 10 & -2 \end{bmatrix}, \) \( D = I_{2 \times 2} \).
(a) Sketch a block diagram for the system.
(b) Determine the system transfer function matrix.
(c) Determine the system poles.
(d) Determine the system transmission zeros \( z_o \). Determine an associated input direction \( u_o \). Show how to determine the associated state direction \( x_o \).
(e) Explain what \( z_o, x_o, u_o \) mean. Hint: If \( x(0) = \cdots \) and \( u(t) = \cdots \), then \( x(t) = \cdots \) and \( y(t) = \cdots \).

EXTRA CREDIT:
Consider the multiple-input multiple-output (MIMO) square LTI system \( L = [A, B, C] \) and let \( S = (I + L)^{-1} \).
(f) Suppose \( z \) is a transmission zero of \( S \) (with directions \( x, u \)). Show that \( z \) must be a pole of \( L \).
(g) Now suppose \( z \) is a pole of \( L \) (with eigenvector \( x \)). Show that \( z \) must be a zero of \( S \). Hint: Must specify directions \( x_o \) and \( u_o \).

Problem 4 (Closed Loop State Space Representation)
Determine a state space representation from \( d_i \) to \( u_p \) for the feedback system with external signals \( (r, d_i, d_o, n) \), state \( x = [x_p \ x_k]^T \), and outputs \( (y, u, e) \) defined by the following equations:
\[
\begin{align*}
    e &= r - y - n \quad u_p = u + d_i \\
    \dot{x}_k &= A_k x_k + B_k e \quad \dot{x}_p = A_p x_p + B_p u_p \\
    u &= C_k x_k + D_k e \quad y = C_p x_p + D_p u_p + d_o.
\end{align*}
\]
with \( D_p = D_k = 0 \). Hint: Just keep what is relevant!