Problem 1 (Gaussian Elimination, Fundamental Subspaces, General Solution, Projections)
Consider the system of linear algebraic equations: \(Ax = b\)
with \(A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 8 & 12 & 16 & 20 \\ 3 & 6 & 9 & 12 & 16 \end{bmatrix}\) and \(b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\).
(a) Parameterize the set of all solutions \(x\).
(b) Determine a basis for each of the four fundamental subspaces: \(\mathcal{R}(A)\), \(\mathcal{R}(A^T)\), \(\mathcal{N}(A)\), \(\mathcal{N}(A^T)\).

HOME:
(c) Assume \(b \in \mathcal{R}(A)\). Determine a solution that lies within \(\mathcal{R}(A^T)\). What can be said about such a solution?
(d) Given that \(b\) does not lie within \(\mathcal{R}(A)\), determine the set of all \(x\) which minimizes the Euclidean norm (distance) \(\|b - Ax\|\). Amongst these, how would you determine the minimum norm solution?

Problem 2 (Controllability and Observability)
Consider the LTI system defined by the state space representation: 
\[\dot{x} = Ax + Bu, \quad y = Cx\]
with \(A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\), \(B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\), \(C = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\), \(D = 0\).
(a) Sketch a block diagram for the system.
(b) Is the system controllable? Explain your answer using multiple tests. Which modes are controllable? uncontrollable? Explain!
(c) Determine a symbolic expression for \(x(t)\) when \(x(0) = 0\).
(d) Determine the set of all states that are reachable from \(x(0) = 0\)?
(e) Is the system observable? Explain your answer using multiple tests. Which modes are observable? unobservable? Explain!
(f) Is \(x(1) = [2 \ 2 \ 0]^T\) reachable from \(x(0) = 0\)? Explain. If so, show how to construct a minimum energy state transferring control law. If not, determine the closest reachable state and then show how to construct a minimum energy state transferring control law.
(g) Is \(x(1) = [2 \ 0 \ 0]^T\) reachable from \(x(0) = 0\)? Explain. If so, show how to construct a minimum energy state transferring control law. If not, determine the closest reachable state and FOR EXTRA CREDIT: show how to construct a minimum energy state transferring control law.

HOME: (h) Determine the form of \(\det(sI - A + BG)\). Can \(A - BG\) possess the eigenvalues 1, 5, 5? What about 5, 5, 5? Explain.

Problem 3 (Modal Analysis)
Consider the linear dynamical system: 
\[\dot{x} = Ax, \quad x(0) = x_o\]
with \(A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & -2 \end{bmatrix}\).
(a) Determine the system’s characteristic equation? (b) Determine all eigenvalues of \(A\). (c) Determine a set of linearly independent eigenvectors for \(A\). Is \(A\) diagonalizable? (d) Compute \(x\).
(e) Determine how each mode can be independently excited via carefully selected (real) initial conditions. Determine the (real) response for each initial condition. HOME: Repeat above for \(\dot{x} = A^T x, x(0) = x_o\).

Problem 4 (State Space Realization and Arithmetic)
(a) Provide two distinct state space realizations for the following LTI system
\[ H(s) = \frac{10s^3}{2s^3 - 1}. \] (1)

Sketch a block diagram for each realization. Indicate all state variables on your diagrams.

**HOME:**

(b) Provide two additional distinct state space realizations for the system above.
(c) Determine a state space representation for the feedback system with external signals \((r, d_i, d_o, n)\), state \(x = [x_p \ x_k]^T\), and outputs \((e, u, y)\) defined by the following equations:
\[
\begin{align*}
e & = r - y - n \\
x_k & = A_k x_k + B_k e \\
u & = C_k x_k + D_k e
\end{align*}
\]
\[
\begin{align*}
u_p & = u + d_i \\
x_p & = A_p x_p + B_p u_p \\
y & = C_p x_p + D_p u_p + d_o.
\end{align*}
\]
Assume that \(M_o \overset{\text{def}}{=} (I + D_p D_k)^{-1}\) and \(M_i \overset{\text{def}}{=} (I + D_k D_p)^{-1}\) are well defined.

**Note:** Do the above for \(D_p = D_k = 0\) first. Then do the non-zero case.

**Problem 5 (Transmission Zeros)** Consider the linear time-invariant (LTI) system:
\[
\begin{align*}
Cx + Du & \text{ with } A = \begin{bmatrix} -2 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -2 & 1 \\
0 & 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 & 100 & 0 \\
-100 & 0 & -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix}.
\end{align*}
\]

(a) Sketch a block diagram for the system. **Note:** \(y_1 = \dot{x}_1 + 100x_3, \ y_2 = \dot{x}_3 - 100x_1\).
(b) Determine the system transfer function matrix. (c) Determine the associated system poles. (d) Determine the associated system transmission zeros and the associated input direction. Show how to determine the state direction.
(e) Use the system block diagram to illustrate the significance of the associated input and state directions. Show how information is propagated through the system.

**DO THE REST AT HOME:**

**Problem 6 (Model Based Compensator)** Consider the linear time-invariant (LTI) plant \(P(s) = \frac{z s}{s + p}\) with \(z, p > 0\), \(A_p = p, B_p = z - p, C_p = 1, D_p = -1\) and variables \((u_p, x_p, y_p)\). Let \(p = 1\) and \(z = 10\).

(a) Design a model based compensator which satisfies the following design specifications: (i) zero steady state error to step reference commands, (ii) closed loop poles at \(s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}\), \(s = -10 \pm j10\).

**Note:** Show how you would determine the final compensator \(K\). (b) Determine the final compensator \(K\). Repeat the above for \(z = 5, 2\). For each of these systems (and the one above), do the following.

(c) Use MATLAB to verify that your design achieves the desired closed loop poles. (d) Let \(L \overset{\text{def}}{=} PK\) denote the open loop transfer function. Use MATLAB to sketch Bode magnitude and phase plots for \(L\). (e) Let \(S = \frac{1}{1+T}, T = 1 - S = \frac{L}{1+L}\). Use MATLAB to plot the magnitude responses \(|S|, |T|, |KS|, |SP|\). (f) Use MATLAB to determine the upward gain margin \(\uparrow GM\), downward gain margin \(\downarrow GM\), phase margin \(PM\), and delay margin \(DM\) for your design? (g) Plot the response (output \(y\) and control \(u\)) to a unit step reference command. Explain why your design achieves the steady state specification. Give the main reason for the overshoot in the output \(y\). How is the response enhanced if you use a command pre-filter \(W = \frac{1}{s^2 + 2z}\) where \(-z\) is a zero of the compensator \(K\). (h) Discuss what happens in (d)-(g) as \(z\) is decreased toward \(p = 1\) from above. What can be said about near right half plane pole-zero cancellations?

**Problem 7 (SVD)** Consider the matrix \(M = \begin{bmatrix} 1 & 1 \\
1 & -1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\
0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\
1 & 1 \end{bmatrix}\). Plot \(y = Mx\) in the \(y_1y_2\) plane when \(x\) lies on the unit circle \(||x|| = 1\). Clearly discuss your plot using SVD concepts.