Problem 1 (Gaussian Elimination, Fundamental Subspaces, General Solution, Projections)
Consider the system of linear algebraic equations:

\[ Ax = b \]

with \( A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix} \) and \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \).

(a) Parameterize the set of all solutions \( x \).
(b) Determine a basis for each of the four fundamental subspaces: \( \mathcal{R}(A), \mathcal{R}(A^T), \mathcal{N}(A), \mathcal{N}(A^T) \).
(c) Determine a solution that lies within the range space of \( A^T \). What can be said about such a solution?
(d) Given that \( b = [b_1 \ b_2 \ b_3 \ b_4 ]^T \) does not lie within the range space of \( A \), determine the set of all \( x \) which minimizes the Euclidean norm (distance) \( \|b - Ax\| \). Amongst these, how would you determine the minimum norm solution?

Problem 2 (State Space Realization and Arithmetic)
(a) Provide two distinct state space realizations for the following LTI system

\[ H(s) = \begin{bmatrix} 20s^4 + 10 \\ 4s^4 + s^2 + 1 \end{bmatrix} + 10. \]  

(b) Determine a basis for each of the four fundamental subspaces: \( \mathcal{R}(A), \mathcal{R}(A^T), \mathcal{N}(A), \mathcal{N}(A^T) \).
(c) Determine a solution that lies within the range space of \( A^T \). What can be said about such a solution?
(d) Given that \( b = [b_1 \ b_2 \ b_3 \ b_4 ]^T \) does not lie within the range space of \( A \), determine the set of all \( x \) which minimizes the Euclidean norm (distance) \( \|b - Ax\| \). Amongst these, how would you determine the minimum norm solution?

Problem 3 (Modal Analysis)
Consider the linear dynamical system:

\[ \dot{x} = Ax \quad x(0) = x_0 \]

with \( A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & -2 \end{bmatrix} \). (a) Determine the system’s characteristic equation? (b) Determine all eigenvalues of \( A \). (c) Determine a set of linearly independent eigenvectors for \( A \). Is \( A \) diagonalizable? (d) Compute \( x \).
(e) Determine how each mode can be independently excited via carefully selected (real) initial conditions. Determine the (real) response for each initial condition.

Problem 4 (State Computation)
Consider the linear system:

\[ \dot{x} = Ax + Bu \quad y = Cx \]
with $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 0 & 2 & 10 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$.

Sketch a block diagram for the system. (b) Compute $x$ and $y$ when $u = \delta(t)$ (unit impulse) and $x(0) = 0$. 

**Problem 5 (Transmission Zeros)**

Consider the linear time invariant (LTI) system:

$$\dot{x} = Ax + Bu \quad y = Cx + Du$$

with $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ -94 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$.

Sketch a block diagram for the system. (b) Determine the system transfer function matrix. (c) Determine the associated system poles. (d) Determine the associated system transmission zeros and the associated input and state directions. (e) Use the system block diagram to illustrate the significance of the associated input and state directions. Show how information is propagated through the system.

**Problem 6 (Controllability and Observability)**

Consider the LTI system defined by the state space representation:

$$\dot{x} = Ax + Bu \quad y = Cx$$

with $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix}$.

Sketch a block diagram for the system. (b) Is the system controllable? Explain your answer using multiple tests. (c) Determine an expression for $x(t)$ when $x(0) = 0$. (d) Determine the set of all states that are reachable from $x(0) = 0$? Is $x(1) = [5 \ 2 \ 2]^T$ reachable from $x(0) = 0$? Explain. If so, show how to construct a minimum energy state transferring control law. Is $x(1) = [2 \ 2 \ 3]^T$ reachable from $x(0) = 0$? Explain. If so, show how to construct a minimum energy state transferring control law. (e) What closed loop poles can be achieved with full state feedback? Explain and then prove your claim.

(f) Is the system observable? Explain your answer using multiple tests. (g) Determine an expression for $y$ when $u(t) = 0$ and $x(0) = x_o$. (h) Determine the set of all possible initial conditions $x_o$. Show several methods for determining $x_o$. (i) Show how you would construct an observer for the above system. What closed loop poles can be achieved via $H$ for $(A - HC)$? Explain and then prove your claim.

**Problem 7 (Model Based Compensator)**

Consider the linear time invariant (LTI) plant

$$P(s) = \frac{z - s}{s - p}$$
with \( z, p > 0, A_p = p, B_p = z - p, C_p = 1, D_p = -1 \) and variables \((u_p, x_p, y_p)\). Let \( p = 1 \) and \( z = 10 \).

(a) Design a model based compensator which satisfies the following design specifications:

(i) zero steady state error to step reference commands,

(ii) closed loop poles at \( s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} \), \( s = -10 \pm j10 \).

Show how you would determine the final compensator \( K \).

**HOME:**

(b) Determine the final compensator \( K \).

Repeat the above for \( z = 5, 2 \). For each of these systems (and the one above), do the following.

(c) Use MATLAB to verify that your design achieves the desired closed loop poles. (d) Let \( L \overset{\text{def}}{=} PK \) denote the open loop transfer function. Use MATLAB to sketch Bode magnitude and phase plots for \( L \). (e) Let \( S = \frac{1}{1+L}, T = 1 - S = \frac{L}{1+L} \). Use MATLAB to plot the magnitude responses \(|S|, |T|, |KS|, |SP|\). (f) Use MATLAB to determine the upward gain margin \( \uparrow GM \), downward gain margin \( \downarrow GM \), phase margin \( PM \), and delay margin \( DM \) for your design? (g) Plot the response (output \( y \) and control \( u \)) to a unit step reference command. Explain why your design achieves the steady state specification. Give the main reason for the overshoot in the output \( y \). How is the response enhanced if you use a command pre-filter \( W = \frac{z}{z+2} \) where \(-z_k\) is a zero of the compensator \( K \). (h) Discuss what happens in (d)-(g) as \( z \) is decreased toward \( p = 1 \) from above. What can be said about near right half plane pole-zero cancelations?