Problem #1

Gaussian Elimination, Fundamental Subspaces, Bases, General Solution, SVD

Consider the linear algebraic system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 2 \\ 6 & 6 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

a) Show how Gaussian Elimination may be used to determine the general solution.

b) From your calculations in (a), show how to obtain a basis for each of the four fundamental subspaces: $\mathrm{R}(A), \mathrm{R}(A^\dagger), \mathrm{N}(A), \mathrm{N}(A^\dagger)$.

c) Using MATLAB verify (using SVD command) that $A = U \Sigma V^H$ where

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} 0.1544 & 0.7222 \\ 0.2705 & -0.6872 \\ 0.4250 & 0.0350 \\ 0.8419 & 0.0700 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} -0.3015 & -0.6030 \\ -0.3015 & -0.6030 \\ 0.8603 & -0.2791 \\ 0.8603 & -0.2791 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} 0.6406 & -0.2995 & 0.7071 \\ 0.6406 & -0.2995 & -0.7071 \\ 0.4235 & 0.9059 & 0 \end{bmatrix}.$$

Show how one can construct the general solution in terms of the $U_i, V_i, \Sigma_i$ and $\mathbf{b}$.

d) Using the above singular value decomposition (SVD) for $A$, relabel the fundamental subspaces $\mathrm{R}(A), \mathrm{N}(A^\dagger)$, $\mathrm{R}(A^\dagger), \mathrm{N}(A)$ to the matrices $U_i, V_i$.

Problem #2

Least Squares, SVD

Using the $A$ and $\mathbf{b}$ given in Problem #1, parameterize the set of all $\mathbf{x}$ which minimize $\| \mathbf{b} - A\mathbf{x} \|_2$. Write your answer in terms of the $U_i, V_i$ given in Problem #1. In your answer, discuss how $\mathbf{b}$ is projected onto $\mathrm{R}(A)$.

Problem #3

Minimum Norm Solution

Using the $A$ and $\mathbf{b}$ given in Problem #1, find the minimum norm solution $\| \mathbf{x} \|_2$ which minimizes $\| \mathbf{b} - A\mathbf{x} \|_2$. In your answer, use the $U_i, V_i$ given in Problem #1. Discuss how the projections which are taking place is how they are obtained.
Problem #4  Systems of Linear Equations: Fundamental Concepts

Consider the general linear algebraic system \( Ax = b \) where \( A \in \mathbb{C}^{m \times n} \), \( b \in \mathbb{C}^{m \times 1} \):

a) Given \( b \in \mathbb{C}^m \), when does a solution exist?
b) When does a solution exist for any \( b \in \mathbb{C}^m \)?
c) Given that a solution exists, when is the solution unique?
d) Assuming a solution does not exist, show how to find a vector \( x \in \mathbb{C}^n \) which minimizes \( \| b - Ax \|_2 \).

When is such a vector unique? Show how to project \( b \) onto \( \mathbb{R}(A) \).

e) Assuming a solution exists, show how to find a minimum norm vector \( x \) which minimizes \( \| x \|_2 \).

Is the vector unique? Discuss all projections which are taking place and how they are computed.
f) When does the matrix \( A \) have a right inverse? When is the right inverse unique?
g) When does the matrix \( A \) have a left inverse? When is the left inverse unique?

Problem #5  Linearization, State Space, Car Dynamics

The dynamics for a car may be approximated by the following second order nonlinear ordinary differential equation:

\[
\ddot{z} = -\frac{k}{m} \dot{z}^2 + \frac{1}{m} F
\]

Suppose \( k = m = 1 \).

a) Linearize the above dynamical equation about the equilibrium \( \dot{z}_0 = v_0 = 1 \), \( F_0 = 1 \).
b) Show how your linear model may be rewritten in state space form:

\[
\begin{align*}
\delta x &= A \delta x + B \delta u \\
\delta y &= C \delta x + D \delta u
\end{align*}
\]

Problem #6  State Space Realization, Block Diagram

Show how the single-input single-output (SISO) transfer function

\[
H(s) = \frac{10s^3 - s + 6}{s^3 + 40s^2 - 20s}
\]

may be synthesized in state space form. Discuss the uniqueness of your realization. Use block diagram ideas in your discussion.
Problem #7  Laplace, Steady State Analysis, Block Diagram, Car Dynamics

Consider the following linearized car dynamics:

\[
\begin{bmatrix}
\dot{x} \\
v
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & -2
\end{bmatrix}
\begin{bmatrix}
x \\
v
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u \\
x(0) = x_0 \\
v(0) = v_0
\]

a) Sketch a block diagram for the above system.

b) Determine an input \( u \) such that \( v(t) = 10 + \sin(t) \) for \( t > 0 \).

c) Use the method of the transfer function to determine an input \( u \) such that the steady-state speed is \( v_{ss} = 10 + \sin(t) \).

Problem #8  Modal Analysis, Eigenvalues, Eigenvectors

a) Obtain a state space representation for the following dynamical system:

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

b) Discuss the three (3) natural modes of the above dynamical system.

Show how each mode may be uniquely excited by appropriately selecting the initial condition.

Problem #9  Controllability & Observability

a) Obtain a state space representation for the following dynamical system:

b) For what values of \( a, b \in \mathbb{R} \) is the system (i) controllable & observable

(iii) controllable & unobservable

(iv) uncontrollable & unobservable.

Back up your discussion with rank tests & eigenvector tests... both!
Problem #10: Controllability, State Transfer

Consider the dynamical system given in Problem #9 with $a = 1$, $b = 0$ and all initial conditions zero.

a) Find a basis for the set of all reachable states.

b) Show how to construct a control which achieves $x(1) = x_f$, assuming $x_f$ is reachable. 
   Hint: Write $x_f$ as a linear combination of eigenvectors. Is your control unique?

c) Suppose $x_f$ is not reachable. Show how to construct a control $u$ so that $\|x(1) - x_f\|_2$ is minimum. Is your answer unique?

Problem #11: Transmission Zeros

a) Determine the transmission zeros and transmission zero directions for the dynamical system considered in Problem #9.

b) Use the block diagram in Problem #9 to show how the transmission zero directions may be visualized.