Problem #1: Markov Parameters, Hankel Matrices

Consider the system
\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

The system possesses a tfm given by
\[ G(s) = C(sI - A)^{-1}B + D. \]

Fact: For sufficiently large \( G(s) \) can be expanded as follows:
\[
G(s) = D + C(sI - A)^{-1}B \\
= D + C \frac{1}{s} \left( I - \frac{A}{s} \right)^{-1}B \\
= D + C \frac{1}{s} \left[ 1 + \frac{A}{s} + \left( \frac{A}{s} \right)^2 + \cdots \right] B \\
= D + \sum_{k=0}^{\infty} \frac{CA^kB}{s^k} \\
= M_0 + \sum_{k=1}^{\infty} \frac{M_k}{s^k}
\]

where
\[ M_0 = D \]
\[ M_k = CA^{k-1}B \quad k = 1, 2, 3, \ldots \]

The matrices \( \{M_k\}_{k=1}^{\infty} \) are referred to as the Markov Parameters for the system \((A, B, C, D)\).

A Hankel matrix is a matrix which is constant along anti-diagonals.

The System Hankel Matrix \( H(A, B, C, D) \) for a state space quad \((A, B, C, D)\) is defined as follows:

\[
H(A, B, C, D) = \begin{bmatrix}
M_1 & M_2 & M_3 & \cdots & M_n \\
M_2 & M_3 & M_4 & \cdots & \vdots \\
M_3 & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
M_n & M_{n+1} & \cdots & \cdots & M_{2n-1}
\end{bmatrix}
= \begin{bmatrix}
CB & CAB & \cdots & CA^{n-1}B \\
CAB & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
CA^{n-1}B & \cdots & \ddots & CA^{2n-2}B
\end{bmatrix}
\]
Note that

\[ \mathcal{H}(A; B; C; D) = \mathcal{O}(C; A) \mathcal{C}(A; B) \]

and that \( \mathcal{H} \) is independent of \( D \).

Let \( x = Qz \) where \( Q \) is a nonsingular change of coordinates.

We then have

\[ \begin{align*}
\hat{z} &= A\hat{z} + Bu \\
y &= C\hat{z} + Du
\end{align*} \]

a) Compute \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \) in terms of \( A, B, C, D, Q \)

b) Show that Markov Parameters are invariant under a nonsingular change of coordinates.

c) Show that the System Hankel Matrix is invariant under a nonsingular change of coordinates.

d) Prove the following theorem

**Theorem:**

\[ \text{rank } \mathcal{H}(A; B; C; D) = n \iff (A; B; C; D) \text{ controllable and observable} \]

**Hint:** Sylvester's Inequality

Assume: \( M \in \mathbb{C}^{m \times n} \), \( N \in \mathbb{C}^{n \times p} \)

Claim: \( \text{rank } M + \text{rank } N - n \leq \text{rank } (MN) \leq \min(\text{rank } M, \text{rank } N) \)
e) The above result implies that

\[ (A;B,C,D) \text{ has a } \mathbf{PZC} \iff \text{rank } H_0(A;B,C,D) < n \]

(i) Use the result to show that

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= 
\begin{bmatrix}
  0 & 0 \\
  1 & -1
\end{bmatrix}
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
+ 
\begin{bmatrix}
  1 \\
  0
\end{bmatrix}
\mathbf{u}
\]

\[
y = [1 -1] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

has a PZC. Note that this Hankel technique does not tell us where the PZC occurs.

(ii) To determine where the PZC occurs we use a PBH technique. Show that the above system has a PZC at \( s = 0 \) by computing

\[
\text{rank } [sI - A \ C]_{s=0}
\]

\[
\text{rank } [sI - A \ B]_{s=0}
\]

Why do we have a PZC? What are we losing?

f) Show that

\[
M_k = \left. \frac{d^{k-1}}{dt^{k-1}} y(t) \right|_{t=0} \quad k=1,2, \ldots
\]

where \( y(t) \equiv C e^{At} B \) is the impulse response matrix of the state space quadruple \((A,B,C,D)\).
Problem #2

Consider the system

\[
\begin{align*}
x' &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

where \( \text{dot}(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \).

Let \( x = Qz \) where \( Q \) is a non-singular change of coordinates.

Given this, we have

\[
\begin{align*}
z' &= A_z z + B_z u \\
y &= C_z z + D_z u
\end{align*}
\]

a) Compute \( A_z, B_z, C_z, D_z \) in terms of \( A, B, C, D \) and \( Q \).

Now assume that \( m = 1 \); i.e., single-input case.

Also assume that \( (A, B) \) is controllable; i.e., \( \text{rank} \ E(A, B) = n \).

b) Now let \( Q = E(A, B) = [B \ AB \ \cdots \ A^{n-1}B] \).

Compute \( A_z, B_z, C_z, D_z \) in terms of the coefficients \( \{a_k\}_{k=0}^{n-1} \) and the Markov parameters \( \{M_k\}_{k=0}^{n} \).

Terminology: The quadruple \( (A_z, B_z, C_z, D_z) \) is called the controllability canonical realization of \( (A, B, C, D) \). Terminology makes sense given our choice of \( Q \).

c) Give necessary and sufficient conditions for a state space quadruple \( (A, B, C, D) \) to have a controllability canonical realization.

d) Compute \( E(A_z, B_z) \). What can you conclude?

e) Do the following dynamical systems have a controllability canonical realization?

(i) \( A = I \), \( B = 0 \), \( C = 1 \)  
Why?

(ii) \( A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( C = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \)  
Why?
f) For what values of $b_0$ and $b_1$ does the following system have a controllability canonical realization?

![System Diagram]

Show your result in 4 distinct ways:

(i) Compute state space triple $(A, B, C)$. Compute \( \text{rank } B(A, B) \).
(ii) Compute \( \text{rank } [\lambda I - A, B] \).
(iii) Find \( y \neq 0 \Rightarrow y^H [\lambda I - A, B] = 0^H \).
(iv) Let \( x_1(t_0) = x_1 \) and \( x_2(t_0) = x_2 \). Compute \( \frac{dx_1(t)}{dt} = b_0 x_2(t) \).

Show that for initial conditions \( x_1 \) and \( x_2 \) at time \( t_0 \) no input \( u(t) \) can transfer the state \( x_1 \) to \( x_2 \) at time \( t \).

9) Let \( n = 4 \). Also let \( p = m = 1 \) (SISO case). Show that

\[
C_{co} (sI - A_{co})^{-1} B_{co} + D_{co} = \frac{n_3 s^3 + n_2 s^2 + n_1 s + n_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} + M_0
\]

where

\[
\begin{bmatrix}
    n_0 \\
    n_1 \\
    n_2 \\
    n_3
\end{bmatrix} =
\begin{bmatrix}
    a_1 & a_2 & a_3 & 1 \\
    a_2 & a_3 & 1 \\
    a_3 & 1
\end{bmatrix}
\begin{bmatrix}
    M_1 \\
    M_2 \\
    M_3 \\
    M_4
\end{bmatrix} = C^{-1}
\begin{bmatrix}
    M_1 \\
    M_2 \\
    M_3 \\
    M_4
\end{bmatrix}
\]

\( C = C(A_{co}, B_{co}) \)

Remember this ordering!
\[
A_c = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-d_0 & -a_1 & -a_2 & -a_3
\end{bmatrix}, \quad B_c = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} - \text{Controller Canonical Form}
\]

\[
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4
\end{bmatrix} = C_c \begin{bmatrix}
n_0 \\
n_1 \\
n_2 \\
n_3
\end{bmatrix}
\]

h) Show that the following block diagram is a visualization for a controllability canonical realization.

1) \( G(s) = 10 \left[ \frac{s^3 + 2s^2 + 2s}{(s+1)(s+1-j)(s+1+j)} \right] \)

1. Compute a 3rd order controllability canonical realization \((A_c, B_c, C_c, D_c)\) for \( G(s) \).

2. Compute \( \text{rank} \left[ sI - A_c \right] \big|_{s=-1 \pm j1} \). Is \((A_c, B_c, C_c, D_c)\) controllable? What can be said about \( s=-1 \pm j1 \) mode?

3. Compute \( \text{rank} \left[ sI - A_c \right] \big|_{s=-1 \pm j1} \). Is \((A_c, B_c, C_c, D_c)\) observable?

4. Does \((A_c, B_c, C_c, D_c)\) possess a PZC?

5. Is \((A_c, B_c, C_c, D_c)\) minimal?

6. Give a minimal order realization for \( G(s) \).
7. Does there exist a matrix $G$ which will place the eigenvalues of $A_{co} - B_{co} G$ at arbitrary locations in the $s$-plane?

8. Compute a matrix $G$ such that $A_{co} - B_{co} G$ has eigenvalues at $s = -\frac{1}{2} \pm j \frac{\sqrt{3}}{2}$. Note that such a $G$ is not unique. I claim that the set of $G$ which yield these eigenvalues spans a one-dimensional subset of $\mathbb{R}^3$. To parameterize this subspace we require one parameter, say $\lambda$. I want you to compute $G = G_\lambda$ where

$$\det (sI - A_{co} + B_{co} G) = (s + \frac{1}{2} - j \frac{\sqrt{3}}{2})(s + \frac{1}{2} + j \frac{\sqrt{3}}{2})(s - \lambda).$$

9. Does there exist a matrix $H$ such that $A_{co} - H C_{co}$ has eigenvalues at $s = 0, 0, 0^3$? Why?

10. What can be said about the eigenvalues of $A_{co} - H C_{co}$ for an arbitrary matrix $H$?
Problem #3

Observability Canonical Form

Consider the system

\[
\begin{align*}
    \dot{x} &= Ax + Bu \\
    y &= Cx + Du
\end{align*}
\]

where \( \det(sI-A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \).

Let \( z = T^T x \) where \( T \) is a non-singular change of coordinates.

Given this, we have

\[
\begin{align*}
    \dot{z} &= A_{ob} z + B_{ob} u \\
    y &= C_{ob} z + D_{ob} u
\end{align*}
\]

a) Compute \( A_{ob}, B_{ob}, C_{ob}, D_{ob} \) in terms of \( A, B, C, D, \) and \( T \).

Now assume that \( p = 1 \); i.e., single-output case.

Also assume that \( (A, C) \) is observable; i.e., \( \text{rank } \Theta(C,A) = n \).

b) Now let \( T = \Theta(C, A) \equiv \left[ \begin{array}{c} C \\ CA \\ \vdots \\ C A^{n-1} \end{array} \right] \). Compute \( A_{ob}, B_{ob}, C_{ob}, D_{ob} \) in terms of the coefficients \( s_k \), \( k = 0 \) and the Markov parameters \( \delta_{k=0}^n \). What is the relationship between \( A_{ob}, B_{ob}, C_{ob}, D_{ob} \) in terms of \( A_0, B_0, C_0, D_0 \) ?

Terminology: The quadruple \( (A_{ob}, B_{ob}, C_{ob}, D_{ob}) \) is called the observability canonical realization of \( (A, B, C, D) \).

Terminology makes sense given our choice of \( T \).

c) Give necessary and sufficient conditions for a state space quadruple \( (A, B, C, D) \) to have an observability canonical realization.

d) Compute \( \Theta(C_{ob}, A_{ob}) \). What can you conclude?

e) Do the following dynamical systems have an observability canonical realization?

(i) \( A = 1 \quad B = 1 \quad C = 0 \) \quad Why?

(ii) \( A = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \) \quad Why?
f) For what values of \( C_0 \) and \( C_1 \) does the following system have an observability canonical realization?

\[
\begin{align*}
U_2 & \quad \lambda & \quad x_1 & \quad C_0 & \quad y \\
\lambda & \quad x_2 & \quad C_1 \\
U_2 & \quad \lambda & \quad x_1 & \quad C_0 & \quad y \\
\end{align*}
\]

Do this in 4 distinct ways:

(i) Compute state space representation \((A;B;C)\). Compute \(\Theta(C;A)\)

(ii) Compute rank \[
\begin{bmatrix}
\lambda I - A \\
C
\end{bmatrix}
\]

(iii) Find \( x \neq 0 \) s.t. \[
\begin{bmatrix}
\lambda I - A
\end{bmatrix} x = 0
\]

(iv) Let \( u_1 = u_2 = 0 \) and \( x(0) = x_1 \) and \( x(0) = x_2 \). Compute \( y(t) \).

What are the set of all initial conditions \( x_1, x_2 \) at time 0 which cannot be determined uniquely from \( y(t) \). Does this result depend on our assumption that \( u_1 = u_2 = 0 \)? Why?

9) Let \( n = 4 \), Also let \( p = m = 1 \) (SISO case). Show that

\[
C_{ob}(sI - A_{ob})^{-1}B_{ob} + D_{ob} = \frac{m_3 s^3 + m_2 s^2 + m_1 s + m_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} + M_0
\]

where

\[
\begin{bmatrix}
m_0 \\
1 \\
m_2 \\
m_3
\end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & 1 \\
a_2 & a_3 & 1 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\
M_2 \\
M_3 \\
M_4
\end{bmatrix} = \Theta_0^{-1} \begin{bmatrix} M_1 \\
M_2 \\
M_3 \\
M_4
\end{bmatrix}
\]

\( \Theta_0 \equiv \Theta(C_0, A_0) \)

Remember this order
\[
A_0 = \begin{bmatrix}
0 & 0 & 0 & -a_0 \\
1 & 0 & 0 & -a_1 \\
0 & 1 & 0 & -a_2 \\
0 & 0 & 1 & -a_3 \\
\end{bmatrix} \quad C_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}
\]

and
\[
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4
\end{bmatrix} = C_0 \begin{bmatrix}
\eta_0 \\
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix}
\]

b) Show that the following block diagram is a visualization for an observability canonical realization.

1) \( G(s) = 100 \left[ \frac{s^3 + 1}{s^3 - s^2 + s} \right] \)

1. Compute a 3rd order observability canonical realization \((A_{ob}, B_{ob}, C_{ob}, D_{ob})\) for \( G(s) \).

2. For what values of \( s \), if any, does \([sI - A_{ob} \ B_{ob}]\) lose row rank? What does this imply?

3. For what values of \( s \), if any, does \([sI - A_{ob}] C_{ob}\) lose column rank? What does this imply?

4. Does \((A_{ob}, B_{ob}, C_{ob}, D_{ob})\) possess a PZC?
5. Is \((A_0, B_0, C_0, D_0)\) minimal?

6. Give a minimal order realization for \(G(s)\)?

7. Does there exist a matrix \(G\) which will place the eigenvalues of \(A_{0b} - B_{0b}G\) at arbitrary locations in the \(s\)-plane?

8. Does there exist a matrix \(G\) which will place an eigenvalue of \(A_{0b} - B_{0b}G\) at \(s = -1\)? Why? What can be said about the eigenvalues of \(A_{0b} - B_{0b}G\) for an arbitrary \(G\)?

9. Does there exist an \(H\) which will place the eigenvalues of \(A_{0b} - H_{0b}C_0\) at arbitrary locations in the \(s\)-plane?

10. Compute the set of all matrices \(H_{\lambda_1, \lambda_2}\) which result in an eigenvalue at \(s = -1\).

   i.e. \(\det(sI - A_{0b} + H_{\lambda_1, \lambda_2}C_0) = (s+1)(s-\lambda_1)(s-\lambda_2)\)
Problem #4  

**Controllability Matrix**

Consider the system  
\[ \dot{x} = Ax + Bu \]  
\[ A \in \mathbb{C}^{n \times n} \]  
\[ B \in \mathbb{C}^{n \times m} \]

**Definition:**  
\[ G^0 = B \]  
\[ G^k = [B \ AB \ldots A^k B] \quad k = 1, 2, 3, \ldots \]

\[ \Rightarrow G(A; B) = G^{n-1} \quad - \text{Controllability Matrix} \]

a) Use the Cayley-Hamilton Theorem to show that \( \text{rank } G(A; B) = \text{rank } G^k \quad \forall k \geq n-1 \)

b) Let \( \Pi = \) degree of the minimal polynomial of \( A \); i.e., the degree of the minimal order monic polynomial \( \Psi(s) \) which annihilates \( A \).

Hint. Let \( \Psi(s) = s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \)

Show that \( \text{rank } G(A; B) = \text{rank } G^k \quad \forall k \geq \Pi \)

c) Show that if \( \text{rank } G^k = \text{rank } G^{k+1} \) then \( \text{rank } G(A; B) = \text{rank } G^k \quad \forall k \geq k \).

d) Assume \( \text{rank } B = r \).

(i) Show that if \( \text{rank } B^r = \text{rank } B^0 = r \), then \( \text{rank } G(A; B) = \text{rank } G^{n-r} = r \)  
Otherwise \( \text{rank } B^1 > r + 1 \)  

(ii) Show that if \( \text{rank } B^2 = \text{rank } B^1 > r + 1 \), then \( \text{rank } G(A; B) = \text{rank } G^{n-r} > r + 1 \)  
Otherwise \( \text{rank } B^2 > r + 2 \)  

(iii) Show that if \( \text{rank } B^3 = \text{rank } B^2 > r + 2 \), then \( \text{rank } G(A; B) = \text{rank } G^{n-r} > r + 2 \)  
Otherwise \( \text{rank } B^3 > r + 3 \)  

\vdots

(iv) Show that if \( \text{rank } G^{n-r} = \text{rank } G^{n-r-1} > r + n - r - 1 = n - 1 \) then \( \text{rank } G(A; B) = \text{rank } G^{n-r} > n - 1 \), otherwise \( \text{rank } G^{n-r} > n \) which implies \( \text{rank } G(A; B) = \text{rank } G^{n-r} = n \) since \( \text{rank } G^{n-r} \leq \text{rank } G(A; B) \leq n \).

The above proves the following important theorem:
Theorem

Assume: $\text{rank } B = r$

Claim: $\text{rank } \mathcal{E}(A;B) = \text{rank } [B\ AB\ \cdots\ A^{n-1}B]$ 

A corollary of the above theorem is as follows:

Corollary: Determining Controllability

Assume: $\text{rank } B = r$

Claim: $(A;B)$ controllable $\iff$ $\text{rank } G^{n-r} = \text{rank } [B\ \cdots\ A^{n-r}B] = n$

$\iff$ $G^{n-r}$ has full row rank

$\iff$ $G^{n-r}$ has no left null space

$\iff$ $G^{n-r}$ is surjective (onto)

$\iff$ $G^{n-r}$ has a right inverse

$\iff$ $G^{n-r}(G^{n-r})^* \text{ is non-singular}$

(i) Use the above rank test on $G^{n-r}$ to determine whether or not the following system is controllable.

Also compute $\text{rank } \mathcal{E}(A;B)$.

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
x &= [x_1\ x_2\ x_3]^T \\
u &= [u_1\ u_2]^T \\
y &= Cx + Du \\
y &\equiv [y_1\ y_2]^T
\end{align*}
\]
(ii) Use the PBH eigenvalue-left eigenvector test to reach a more specific conclusion.

(iii) Is the above system observable?

(iv) Compute the tfm for the above system: \( G(s) = C(sI-A)^{-1}B + D \).

(v) Compute the zeros of the system.

(vi) Does the above system possess a PEC? Why?

(vii) Is the above system stabilizable via state feedback?

f) Suppose \( \text{rank} \ B = r < m \). Is \( (A, B) \) necessarily uncontrollable? The answer is \text{HELL NO}! Give a counterexample.

\text{Moral:} Just because we experience one engine failure during a flight doesn't mean that we should panic!
Problem #5  

**Canonical Form**

Consider the system

\[
\begin{align*}
\dot{x} & = Ax + Bu \\
y & = Cx + Du
\end{align*}
\]

**Assume:** \( \text{rank } \mathcal{E}(A; B) = n_1 < n \)

Let \( x = Qz \) where

\[
Q = [q_1 \ldots q_{n_1} q_{n_1+1} q_{n_1+2} \ldots q_n]
\]

\( q_1, \ldots, q_{n_1} \) are any \( n_1 \) LI columns of \( \mathcal{E}(A; B) \)

\( q_{n_1+1}, q_{n_1+2}, \ldots, q_n \) are any \( n - n_1 \) vectors which make \( Q \) non-singular.

We then get

\[
\begin{align*}
\dot{z} & = A_c z + B_c u \\
y & = C_c z + D_c u
\end{align*}
\]

(a) Compute \( A_c, B_c, C_c, D_c \) in terms of \( A, B, C, D, Q \).

(b) Show that this new system may be written as follows:

\[
\begin{align*}
\begin{bmatrix} \dot{z}_c \\ \dot{z}_e \end{bmatrix} & = \begin{bmatrix} A_c & n_1 \rightarrow n \\ O_{(n-n_1) \times n_1} & A_e \end{bmatrix} \begin{bmatrix} z_c \\ z_e \end{bmatrix} + \begin{bmatrix} B_c \\ O_{(n-n_1) \times 1} \end{bmatrix} u \\
y & = \begin{bmatrix} C_c & C_e \end{bmatrix} \begin{bmatrix} z_c \\ z_e \end{bmatrix} + D u
\end{align*}
\]

(c) Show that the system

\[
\begin{align*}
\dot{z}_c & = A_c z_c + B_c u \\
y & = C_c z_c + D u
\end{align*}
\]
\( p \) is controllable.

d) Show that \( A_c \) contains all of the controllable modes of the system \((A_c, B_c, C_c, D_c)\) and that \( A_c \) contains all of the uncontrollable modes.

e) Show that \( C(sI-A)^{-1}B + D = C_c \bar{C}(sI-A_c)^{-1}B_c + D_c = C_c(sI-A_c)^{-1}B_c + D \).

**Visualization:**

\[
\begin{align*}
&u \\
&\rightarrow B_c \\
&\rightarrow \bar{z}_c \\
&\rightarrow \frac{1}{s} \\
&\rightarrow A_c \\
&\rightarrow \bar{z}_c \\
&\rightarrow \frac{1}{s} \\
&\rightarrow A_c \\
&\rightarrow D \\
&\rightarrow z_c \\
&\rightarrow z_c \\
&\rightarrow C_c \\
&\rightarrow y
\end{align*}
\]

**Note:** \( u \) has no effect on \( \bar{z}_c \); i.e. \( \bar{z}_c \) is uncontrollable.

**Terminology:** The state space quadruple \((A_c, B_c, C_c, D_c)\) is called the \textbf{\( EE \) Canonical Form} for \((A, B, C, D)\).
f) Consider the differential equation
\[ \dddot{y} + 2 \ddot{y} + 2\dot{y} = 10 \ddot{u} + 20\dot{u} + 20u \]

(i) Compute a 3rd order observability canonical realization \((A, B, C, D)\) for this differential equation.

(ii) Is the resulting realization controllable?

(iii) Transform the realization to \(\tilde{G}\) form; i.e., determine a valid \(Q\) and then compute \(A, A\bar{c}, A\bar{c}, B, C, C, C, C, D\).
Are they unique?

9) Prove the following very important theorem.

**Theorem:** (PBH Left Eigenvector Test)

\[(A, B) \text{ uncontrollable} \iff \exists \lambda \in \mathbb{C} \; \exists y \neq 0 \; \exists y^H [\lambda I - A \; B] = 0^H\]

**Def:** \((A, B)\) is stabilizable if all unstable modes of \(A\) are controllable.

\(\lambda\) is unstable if \(\text{Re}\lambda \geq 0\).

h) Prove that

\[(A, B) \text{ is stabilizable} \iff \text{rank } [sI - A \; B] = n \]

\[\forall s \in \overline{C_+} = \{ s \in \mathbb{C} \mid \text{Re}\ s \geq 0 \} \]
Problem #6  Canonical Form

Consider the system
\[ \dot{x} = Ax + Bu \quad A \in \mathbb{C}^{n \times n} \quad B \in \mathbb{C}^{n \times m} \]
\[ y = Cx + Du \quad C \in \mathbb{C}^{p \times n} \quad D \in \mathbb{C}^{p \times m} \]

Assume: \( \text{rank } \Theta_c(A) = n_1 < n \)

Let \( z = T \cdot x \) where
\[
T = \begin{bmatrix}
- t_1 \\
- t_2 \\
- t_3 \\
\vdots \\
- t_{n_1+1} \\
- t_n \\
\end{bmatrix}
\]

\( t_1, \ldots, t_{n_1} \) are any \( n_1 \) LI rows of \( \Theta_c(A) \)
\( t_{n_1+1}, \ldots, t_n \) are any \( n-n_1 \) vectors
which make \( T \) non-singular.

We then get
\[
\dot{z} = A_\infty z + B_\infty u \\
y = C_\infty z + D_\infty u
\]

a) Compute \( A_\infty, B_\infty, C_\infty, D_\infty \) in terms of \( A, B, C, D, T \).
b) Show that this new system may be written as follows:
\[
\begin{align*}
\dot{z} & \quad n \times 1 \\
n-n, \quad [z] & \quad \begin{bmatrix} A_0 & A_{n_1} \\ A_{n_0} & A_0 \end{bmatrix} [z_0] + [B_0] u \\
\quad p \times 1 \\
\quad y & \quad [C_0 \quad 0] [z_0] + D u
\end{align*}
\]
c) Show that the system

\[ \dot{z}_0 = A_0 z_0 + B_0 u \]

\[ y = C_0 z_0 + D u \]

is observable.

d) Show that \( A_0 \) contains all of the observable modes of the system \((A_0, B_0, C_0, D_0)\) and that \( A_0 \) contains all of the unobservable modes.

e) Show that

\[ c(sI - A_0^{-1}B + D) = C_0 (sI - A_0^{-1})^{-1}B_0 + D_0 \]

\[ = C_0 (sI - A_0)^{-1}B_0 + D \]

**Visualization:**

![Diagram](image)

**Note:** \( y \) does not see \( z_0 \), i.e., \( z_0 \) is unobservable.

**Terminology:** The state space quadruple \((A_0, B_0, C_0, D_0)\) is called the Canonical Form for \((A, B, C, D)\).
f) Consider the differential equation
\[ \dddot{y} + 2\ddot{y} + 2y = 10\ddot{u} + 30\dot{u} + 40u + 20u \]

(i) Compute a 3rd order controllability canonical realization \((A_1B_1C_1D_1)\) for this differential equation.

(ii) Is the realization observable?

(iii) Compute a \(\Theta\Theta\) realization for \((A_1B_1C_1D_1)\).

9) Prove the following very important theorem.

**Theorem:** (PBH Right Eigenvector Test)

\[ (A_1C) \text{ unobservable } \iff \exists \lambda \in \mathbb{C} \exists x \neq 0 \in \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} x = 0 \]

**Defn:** \((A_1C)\) is detectable if all unstable modes of \(A\) are observable.

\(\lambda\) is unstable if \(\text{Re}\lambda \geq 0\).

h) Prove that

\[ (A_1C) \text{ is detectable } \iff \text{rank } \begin{bmatrix} sI - A \\ C \end{bmatrix} = n \]

\[ \forall s \in \mathbb{C}_+ \equiv \{s \in \mathbb{C} \mid \text{Re}s \geq 0\} \]

CRHP-Closed right half-plane
Problem # 7

**State Feedback Properties**

Consider the open loop system:

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

subject to the state feedback law:

\[u = -Gx + v\]

The resulting closed loop system is then given by:

\begin{align*}
\dot{x} &= (A-BG)x + Bv \\
y &= (C-DG)x + Dv
\end{align*}

**Visualization:**

![Diagram](attachment:image.png)

Show that

a) **Controllability properties are invariant under state feedback**

i.e. \((A,B)\) controllable \(\iff\) \((A-BG,B)\) controllable

**Hint:** Use definition of controllability!
b) Show via an example that

Observability properties are not invariant under state feedback.

i.e. \( (A, C) \) observable \( \implies \) \( (A-BG, C-DG) \) observable

c) Show that the open loop zeros are closed loop zeros and vice versa.

Hint: \[
\begin{bmatrix}
sI-A+BG & -B \\
C-DG & D
\end{bmatrix} =
\begin{bmatrix}
sI-A & -B \\
C & D
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
-G & I
\end{bmatrix}
\]
Problem #8: Output Feedback Properties

Consider the open loop system:
\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

subject to the constant output feedback:
\[
u = -Ky + r
\]

The resulting closed loop system is given by:
\[
\dot{x} = A_{cl} x + B_{cl} r \\
y = C_{cl} x + D_{cl} r
\]

Visualization:

\[\begin{array}{c}
\text{D} \\
\text{B} \\
\frac{1}{3} \\
\text{A} \\
K
\end{array}\]

r \rightarrow u \rightarrow x 

a) Compute \( A_{cl}, B_{cl}, C_{cl}, D_{cl} \) in terms of \( A, B, C, D, K \).

(Assume \((I+KD)^{-1}\) exists)

b) Show that

Controllability Properties are Invariant Under Constant Output Feedback.

i.e., \((A,B)\) controllable \(\iff (A_{cl}, B_{cl})\) controllable.
c) Show that

Observability Properties are Invariant Under Constant Output Feedback

i.e. \((A;C)\) observable \(\iff\) \((A;C;C;D)\) observable

d) Suppose that \((A;B;C;D)\) has no PZC. Can we use \(K\) so that \((A;B;C;D;K)\) has a PZC? Why?