Problem #1  

State Feedback, Pole-Placement, Controller Canonical Form

Consider the open loop system (OLS)
\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]
\[x(0) = x_0 \quad x \in \mathbb{C}^n \quad u \in \mathbb{C}^m \]

**Visualization of Open Loop System:**

\[\text{Diagram of OLS}\]

Consider the following state-feedback control law:
\[
u(t) = -Gx(t) + V(t)\]
\[G \quad \text{Control Gain Matrix}\]

The resulting closed loop system (CLS) may be visualized as follows:

**Visualization of Closed Loop System:**

\[\text{Diagram of CLS}\]
a) Give a state space representation \((A_{Cl}, B_{Cl}, C_{Cl}, D_{Cl})\) for the CLS from \(v\) to \(y\).

b) Show that for the above CLS, we have

\[
x(t) = e^{(A-BG)(t-t_0)} x_0 + \int_{t_0}^{t} e^{(A-BG)(t-\tau)} B v(\tau) \, d\tau.
\]

Note: This equation shows that the Control Gain Matrix \(G\) may be used as a design parameter to modify the natural modes of the OLS. We will show how this is done.

c) Compute the transfer function matrix (TFM) from \(v\) to \(y\).

This computation should offer further support to claim made in the above note.

**NOW let** \(m=1\); i.e., the open loop system is a single-input system.

d) Let \(Q\) be a nonsingular matrix. Let \(x = Qz\) in (a).

Show that

\[
\ddot{z}(t) = (A_c - B_c G_c) z(t) + B_c v(t)
\]

\[
y(t) = (C_c - D_c G_c) z(t) + D_c v(t)
\]

where

\[
A_c = Q^{-1} A Q \quad B_c = Q^{-1} B \quad C_c = C Q \quad D_c = D \quad G_c = G Q
\]

e) Let

\[
A_c = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}
\end{bmatrix} \quad B_c = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]
where \( \det(sI - A) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \) is the characteristic polynomial for \( A \). The pair \((A_c, B_c)\) is referred to as the **Controllable Canonical Form** realization of the pair \((A, B)\).

Show that \[ Q = C C_c^{-1} \quad \leftarrow \text{REMEMBER} \]

where

\[
C = C(A, B) = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}_{n \times (nm)}
\]

and

\[
C_c = C_c^{-1}(A_c, B_c) = \begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} & 1 \\ a_2 & \vdots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{n-1} & \vdots & \ddots & a_2 & 1 \\ 1 & \ldots & \ldots & \ldots & 1 \end{bmatrix}
\]

(See HW#5 Problem #4)

Here, \( C(A, B) \) is the **controllability matrix** for the state space quadruple \((A, B, C, D)\).

Later we shall see that

\[
\text{The system } (A_c, B_c, C, D) \quad \iff \quad \text{rank } C(A, B) = n \quad (\text{full row rank})
\]

Note that this is a "surjectivity" condition; i.e. a condition which guarantees "existence".

Similarly, \[ C_c(A_c, B_c) = [B_c A_c B_c \ldots A_c^{n-1}B_c]_{n \times (nm)} \] is the controllability matrix for the state space quadruple \((A_c, B_c, C_c, D_c)\).

**Note:** You need not show that the inverse of \( C_c(A_c, B_c) \) is the matrix

\[
\begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} & 1 \\ a_2 & \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{n-1} & \vdots & \ddots & a_2 & 1 \\ 1 & \ldots & \ldots & \ldots & 1 \end{bmatrix}
\]

Doing so is just algebra!
Such a matrix which is constant along the anti-diagonals is referred to in the control mathematics literature as a **Hankel Matrix**.

**Note:** The above transformation to Controller Canonical Form can be done if and only if the system \((A,B,C,D)\) is controllable. Only then can we guarantee that \(Q\) is invertible.

f) Let \(\det(sI - A + BG) = s^n + \hat{a}_{n-1} s^{n-1} + \ldots + \hat{a}_1 s + \hat{a}_0\) be the desired closed loop characteristic polynomial.

It has the desired closed loop eigenvalues.

Show that

\[
G = \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix}
= \begin{bmatrix} \hat{a}_0 - a_0 & \hat{a}_1 - a_1 & \cdots & \hat{a}_{n-1} - a_{n-1} \end{bmatrix} C G^{-1}
\]

This is known in the control literature as the **Bass-Gura formula**.

The formula shows that the closed loop poles may be placed anywhere in the complex plane when the OLS is controllable.

This is truly a remarkable result!

9) Let \(A = \begin{bmatrix} -1 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}\).

Is system controllable?

Compute \(G = [g_1, g_2, g_3] \in \mathbb{R}^{1 \times 3} \Rightarrow \lambda(A - BG) = \{0, 0, 0\}\).

How do we know that such a \(G\) exists?
Consider the single-input single-output (SISO) transfer function:

\[
H(s) = \left. \frac{Y(s)}{U(s)} \right|_{s \to \infty, \text{no initial conditions}} = \frac{c_4 s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0}{1 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}
\]

\[
= d + \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}
\]

The above transfer function may be visualized as follows:

\[ u \]

\[ b_0 \]

\[ b_1 \]

\[ b_2 \]

\[ b_3 \]

\[ d \]

\[ \frac{1}{s} \]

\[ -a_0 \]

\[ -a_1 \]

\[ -a_2 \]

\[ -a_3 \]

\[ \frac{1}{s} \]

\[ \frac{1}{s} \]

\[ \frac{1}{s} \]

\[ \frac{1}{s} \]

\[ y \]

\[ z_1 \]

\[ z_2 \]

\[ z_3 \]

\[ z_4 \]

\[ a_0, a_1, a_2, a_3 \]

\[ c_0, c_1, c_2, c_3, c_4 \]

\[ a_0, a_1, a_2, a_3 \]

\[ \dot{z} = A_0 z + B_0 u \]

\[ y = C_0 z + D_0 u \]

\[ \text{i.e. compute them.} \]

Moreover, note that \( A_0 = A_c^T \), \( B_0 = C_c^T \), \( C_0 = B_c^T \), \( D_0 = D_c \).
where \((A_c, B_c, C_c, D_c)\) is a **Controller Canonical Form**.

The resulting state space quadruple \((A_0, B_0, C_0, D_0)\) is called the **Observer Canonical Realization** for \(G(s)\).

Reasons for this terminology will become apparent as we proceed in our study of "observability."

We define the **observability matrix** of a state space quadruple \((A, B, C, D)\) as follows:

\[
\Theta = \Theta(C, A) = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]

where \(A \in \mathbb{C}^{m \times n}\).

Given this, it follows that

\[
\Theta_0 = \Theta(C_0, A_0) = \begin{bmatrix}
C_0 \\
C_0 A_0 \\
C_0 A_0^2 \\
C_0 A_0^3
\end{bmatrix}
\]

Show that

\[
\Theta(C_0, A_0) = G^T(A_0^T, C_0^T) = G^T(A_c, B_c)
\]

and that

\[
\begin{bmatrix}
a_1 & a_2 & a_3 & 1 \\
a_2 & a_3 & 1 \\
a_3 & 1 \\
1
\end{bmatrix}
\]

\[
\Theta_0 = I_{4 \times 4}
\]

and hence that

\[
\Theta_0^{-1} = \begin{bmatrix}
a_1 & a_2 & a_3 & 1 \\
a_2 & a_3 & 1 \\
a_3 & 1 \\
1
\end{bmatrix}
\]
Termology: A matrix which is constant along the anti-diagonals is called a Hankel Matrix.

$\Theta^{-1}$ is an example of a Hankel Matrix.

Later we'll show that

A system $(A, B, C, D)$ is observable $\iff$ rank $\Theta (c, A) = n$ (full column rank)

Note that this is an "infectivity" condition; i.e., a condition which guarantees "uniqueness" given "existence".

4. Given this argue why the quadruple $(A_0, B_0, C_0, D_0)$ is necessarily observable.

2. Show that the quadruple $(A_0, B_0, C_0, D_0)$ is not necessarily controllable. (Give a counterexample)

4. Similarly, argue that the controller canonical quadruple $(A_c, B_c, C_c, D_c)$ is necessarily controllable; show, via a counterexample, that it is not necessarily observable.

9. What pole-zero phenomenon is common to the counterexamples you have constructed in 4 & 6?
Consider the system
\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
y(t) &= C x(t) + D u(t)
\end{align*}
\]
where \( A \in \mathbb{C}^{n \times n} \), \( B \in \mathbb{C}^{n \times m} \), \( C \in \mathbb{C}^{1 \times n} \), and \( D \in \mathbb{C}^{1 \times m} \).

Let \( z = T^T x \) where \( T \) is a nonsingular state transformation to be chosen such that
\[
\begin{align*}
\dot{z}(t) &= A_o z(t) + B_o u(t) \\
y(t) &= C_o z(t) + D_o u(t)
\end{align*}
\]
where
\[
A_o = \begin{bmatrix}
0 & 0 & 0 & -a_0 \\
1 & 0 & 0 & -a_1 \\
0 & 1 & 0 & -a_2 \\
0 & 0 & 1 & -a_3
\end{bmatrix} = A_c^T
\]
\[\text{where } \det(sI - A) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0\]
\[
C_o = \begin{bmatrix}
0 & 0 & 0 & 0 & 1
\end{bmatrix} = B_c^T
\]

Show that \( A_o T = T A \) and \( C_o T = C \).

Let \( T = \begin{bmatrix}
t_1 \\ t_2 \\ t_3 \\ t_4
\end{bmatrix} \).

Show that
\[
\begin{align*}
t_4 &= c \\
t_3 &= a_3 c + c A \\
t_2 &= a_2 c + a_3 c A + c A^2 \\
t_1 &= a_1 c + a_2 c A + a_3 c A^2 + c A^3
\end{align*}
\]
Show that
\[ T = \Theta^{-1}(c_0, A_0) \Theta(C, A) \]

where
\[ \Theta \equiv \Theta(C, A) \equiv \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \]

and
\[ \Theta^{-1}_o \equiv \Theta^{-1}_o(c_0, A_0) = \begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ a_2 & a_3 & 1 \\ a_3 & 1 \\ 1 \end{bmatrix} \]

Note that \( T \) is invertible if and only if \( \Theta \) is invertible, i.e. if and only if the quadruple \((A, B, C, D)\) is observable.

Show that
\[ \Theta(C, A) = C(A^T, C^T) \]

Let
\[ T(C, A) \equiv \Theta^{-1}(c_0, A_0) \Theta(C, A) \]
\[ Q(A, B) \equiv C(A, B) C^{-1}(A_c, B_c) \]

Show that
\[ T(C, A) = Q(A^T, C^T) \]

We have \( A = T^{-1} A_0 T \) \& \( C = C_0 T \). Let \( H \in \mathbb{C}^{n \times 1} \).

If \( A - H C = T^{-1} (A_0 - H_0 C_0) T \), what is \( H \) in terms of \( T \) \& \( H_0 \)?
Suppose we want to choose $H$ so that $A - HC$ has certain "desirable" eigenvalues.

Let $\det(sI - A + HC) = s^n + \hat{a}_{n-1}s^{n-1} + \ldots + a_1s + a_0$ be the desired characteristic polynomial of $A - HC$.

Show that

$$H = \Theta^{-1} \Theta_0 \left[ \begin{array}{c} \hat{a}_0 - a_0 \\ \hat{a}_1 - a_1 \\ \vdots \\ \hat{a}_{n-1} - a_{n-1} \end{array} \right]$$

where

$$\Theta = \Theta(c_1A) = \left[ \begin{array}{c} C \\ CA \\ \vdots \\ CA^{n-1} \end{array} \right]$$

$$\Theta_0 = \Theta(c_0A_0) = \left[ \begin{array}{c} c_0 \\ c_0A_0 \\ \vdots \\ c_0A_0^{n-1} \end{array} \right]$$

will achieve the objective.

Compare the above formula with the Bass-Gura formula.

Let $A = \left[ \begin{array}{ccc} -1 & -2 \\ -3 & \end{array} \right]$, $C = \left[ \begin{array}{ccc} 1 \\ -2 \\ 1 \end{array} \right]$.

Is system observable?

Compute $H = \left[ \begin{array}{ccc} h_1 \\ h_2 \\ h_3 \end{array} \right] \in \mathbb{R}^{3 \times 1}$ so $\lambda(A - HC) = \{0, 0, 0\}$?

How do we know that such an $H$ exists?
Problem #3

MIMO Transmission Zeros

Consider the multi-input multi-output (MIMO) system

\[ \begin{align*}
x(t) &= A x(t) + B u(t) \quad x \in \mathbb{C}^n \quad u \in \mathbb{C}^m \\
y(t) &= C x(t) + D u(t) \quad y \in \mathbb{C}^p
\end{align*} \]

**Defn:**

\((A, B, C, D)\) has a transmission zero at \(z_0\) if \(\exists \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \neq 0 \in \mathbb{C}^{n+m} \) if

\[ u(t) = e^{z_0 t} u_0 \]

and

\[ x(0) = x_0 \]

then

\[ x(t) = e^{z_0 t} x_0 \]

and

\[ y(t) = 0 \quad \forall t \geq 0 \]

Input Frequency Absorption Concept

**Thm:**

\((A, B, C, D)\) has a transmission zero at \(z_0\) if and only if

\[ \exists \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \neq 0 \in \mathbb{C}^{n+m} \]

\[ \begin{bmatrix} z_0 I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

(a) Prove the above theorem.

**Hint:** For \(U(s) = \frac{U_0}{s-z_0}\) we have

\[ (sI-A)X(s) = x_0 + B U_0 \frac{1}{s-z_0} \]

\[ = \frac{1}{s-z_0} \left\{ sx_0 - Ax_0 - z_0 x_0 + Ax_0 + Bu_0 \right\} \]

or

\[ X(s) = \frac{x_0}{s-z_0} + \frac{(sI-A)^{-1}}{s-z_0} \left\{ -(z_0 I-A) x_0 + Bu_0 \right\} \]

\[ Y(s) = \frac{C x_0}{s-z_0} + \frac{C(sI-A)^{-1}}{s-z_0} \left\{ -(z_0 I-A) x_0 + Bu_0 \right\} + D U_0 \frac{1}{s-z_0} \]
Cor: Assume: \( p = m \); i.e., system is square.

Claim: The following are equivalent.

(i) \((A_1 B_1 C_1 D)\) has a transmission zero at \( z_0 \)

(ii) \( [\begin{bmatrix} x_0^0 \\ u_0^0 \end{bmatrix}] = \begin{bmatrix} z_0 I - A & -B \\ C & D \end{bmatrix} [\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}] = [0]

(iii) \( \exists \begin{bmatrix} y_0^H \\ w_0^H \end{bmatrix} = \begin{bmatrix} y_0^H \\ w_0^H \end{bmatrix} \) s.t. \( \begin{bmatrix} z_0 I - A & -B \\ C & D \end{bmatrix} = [0^H \\ 0^H]

(iv) \( \det \begin{bmatrix} z_0 I - A & -B \\ C & D \end{bmatrix} = 0 \)

B Prove the above corollary.

Explain exactly where and why squareness assumption is needed.

C Show that when \( A_1 \) is nonsingular, we have

\[ \det \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \det A_1 \det \begin{bmatrix} A_2 - A_3 A_1^{-1} A_2 \end{bmatrix} \]

Hint:

\[
\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} I & A_2 \\ A_3 A_1^{-1} & A_4 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix}
\]

\[ \det \begin{pmatrix} I & A_2 \\ A_3 A_1^{-1} & A_4 \end{pmatrix} = \det \left( \begin{pmatrix} I & A_2 \\ A_3 A_1^{-1} & A_4 \end{pmatrix} \begin{pmatrix} I & -A_2 \\ 0 & I \end{pmatrix} \right) \]

Thm:

Assume: \( G(s) = C(sI - A)^{-1} B + D \) is a square tfm

Claim: \((A_1 B_1 C_1 D)\) has a transmission zero at \( z_0 \) \( \iff \det (z_0 I - A) \det G(z_0) = 0 \)
Prove the above theorem.

Let \( G(s) = \frac{s-1}{(s+1)(s^2)} \). Obtain an observer canonical realization \((A_0, B_0, C_0, D_0)\) for \( G(s) \).

Show that \( \det \begin{bmatrix} zI-A & -B_0 \\ C_0 & D_0 \end{bmatrix} \bigg|_{z=+1} = 0 \).
Problem # 4


Consider the dynamical system

\[ \dot{x}(t) = A x(t) + B u(t) \quad x \in \mathbb{C}^n \quad u \in \mathbb{C}^m \]
\[ y(t) = C x(t) + D u(t) \quad y \in \mathbb{C}^p \]

\[ \Phi(A; B) \equiv \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \quad \text{Controllability Matrix} \]
\[ \Theta(C; A) \equiv \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \text{Observability Matrix} \]

**Facts:**
1. \((A; B; C; D)\) controllable \(\iff\) \(\text{rank } \Phi(A; B) = n\)
2. \((A; B; C; D)\) observable \(\iff\) \(\text{rank } \Theta(C; A) = n\)

Let \( x = Qz \) where \( Q \) is a nonsingular state transformation.
This coordinate transformation results in

\[ \dot{z}(t) = \hat{A} z(t) + \hat{B} u(t) \]
\[ y(t) = \hat{C} z(t) + \hat{D} u(t) \]

Compute \( \hat{A}; \hat{B}; \hat{C}; \hat{D} \) in terms of \( A; B; C; D; Q; Q^{-1} \).

Compute \( \Phi(\hat{A}; \hat{B}) \) in terms of \( \Phi(A; B) \circ Q \).

Show that controllability is invariant under a similarity transformation.
I.e. \((A; B; C; D)\) controllable iff \((A; B; C; D)\) controllable.

Compute \( \Theta(\hat{C}; \hat{A}) \) in terms of \( \Theta(C; A) \circ Q \).

Show that observability is invariant under a similarity transformation.
I.e. \((A; B; C; D)\) observable iff \((A; B; C; D)\) observable.
9. Assume \( m = p \); i.e. that the system is square.

Show that

\[
\det \begin{pmatrix} sI - \hat{A} & -\hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} = \det \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}
\]

Hint: \( \det \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \det A_1 \det \begin{pmatrix} A_4 - A_3 A_1^{-1} A_2 \end{pmatrix} \)

9. Show that transmission zeros are invariant under similarity transformations.

10. Consider the system

\[
\begin{align*}
\dot{x} &= Ax + B e \\
y &= Cx \\
e &= -Ky + r
\end{align*}
\]

Closed loop system

Visualization:

\[
\begin{array}{c}
r \\
\downarrow \\
e \\
\downarrow \\
G(s) \\
\downarrow \\
y \\
\downarrow \\
K
\end{array}
\]

Show that \( DLS \) and \( CLS \) have the same zeros.

Hint: \( \begin{pmatrix} sI - A + BK & -B \\ C & 0 \end{pmatrix} = \begin{pmatrix} sI - A & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -KC & I \end{pmatrix} \)

Comments:

i) Note that this result is unchanged even when \( K \) is a function of \( s \).

ii) Result holds even if \( G \) is \( K \) have "D" matrices. In such case we need \( I + K(\infty)G(\infty) \) to be nonsingular so that \( I + K(s)G(s) \) is invertible as a tfm.
Problem #5  Popov - Beliehev - Hautus Controllability & Observability Tests

\[ \text{Def: } C(A, B) = [ B \ A B \cdots A^{n-1} B ] \quad \text{Controllability Matrix; } A \in \mathbb{C}^{n \times n} \]

Fact: \( (A, B, C, D) \) controllable \iff rank \( C(A, B) = n \) (full row rank)
\[ \uparrow \text{surjectivity (existence)} \]
\[ \text{condition} \]

Thm: (PBH)

\[ (A, B, C, D) \text{ uncontrollable } \iff \exists y \neq 0 \in \mathbb{C} \quad \text{exists } z \in \mathbb{C} \quad y^H [ I - A - B ] = 0^H \]

\[ \text{a) Prove } (\iff), \text{ we'll prove } (\Rightarrow) \text{ later!} \]

\[ \text{ Hint: } \]
\[ y^H B = 0^H \]
\[ y^H A = z y^H \]
\[ y^H AB = z y^H B = 0^H \]

\[ \text{Def: } \Theta(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \text{Observability Matrix; } A \in \mathbb{C}^{n \times n} \]

Fact: \( (A, B, C, D) \) observable \iff rank \( \Theta(C, A) = n \) (full column rank)
\[ \uparrow \text{injectivity (uniqueness)} \]
\[ \text{condition} \]

Thm: (PBH)

\[ (A, B, C, D) \text{ unobservable } \iff \exists x \neq 0 \in \mathbb{C} \quad \text{exists } z \in \mathbb{C} \quad [ zI - A ] x = 0 \]

\[ \text{b) Prove } (\iff) . \]
Consider the system

\[
\begin{align*}
&u \rightarrow \frac{s+1}{s} \quad H_1 \\
&\quad \rightarrow \frac{1}{s+1} \quad H_2 \\
&\quad \rightarrow \frac{s+1}{s+2} \quad H_3 \\
&\quad \rightarrow y
\end{align*}
\]

(a) Realize \( H_1, H_2, H_3 \) in controller canonical form \((A_i, B_i, C_i, D_i)\) \(i=1,2,3\).

(b) Find a state-space representation \((A, B, C, D)\) for the entire system from \(u\) to \(y\) with state \(x=[x_1 \ x_2 \ x_3]^T\).

(c) Use (a) to show that \(s=-1\) is uncontrollable. Do this in two ways:

(i) Use rank condition

(ii) Use PBH Test

(d) Use (b) to show that \(s=-1\) is unobservable. Do this in two ways:

(i) Use rank condition

(ii) Use PBH Test

(e) Compute \(\det(sI-A)\). What are the system poles?

(f) Compute \(\det\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}\). What are the system zeros?

(g) Why is the \(s=-1\) uncontrollable? Unobservable?
Problem #6

Least Squares

Assume:
1. $\langle x, y \rangle \equiv x^H y$
2. $\|x\| \equiv \sqrt{\langle x, x \rangle}$
3. $A \in \mathbb{C}^{m \times n}$, $b \in \mathbb{C}^m$ - given

We want to solve the following problem:

$$\min_{x \in \mathbb{C}^n} \|b - Ax\|$$

Facts:
1. A minimizer $x_0 \in \mathbb{C}^n$ always exists but is not necessarily unique.
2. $b - Ax_0$ is orthogonal to $\mathbb{R}(A)$

Visualization:

```
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Visualization of orthogonal projection.}
\end{figure}
```

Defn:

- $P = \mathbb{C}^m \rightarrow \mathbb{R}(A)$
- $b \rightarrow Pb \equiv Ax_0$
- Projection of $\mathbb{C}^m$ onto $\mathbb{R}(A)$

Show that $x_0$ satisfies the Normal Equations:

$$A^H A x_0 = A^H b$$
b) Show that \( \mathcal{R}(A^H A) = \mathcal{R}(A^H) \)

Hints: \( \mathcal{R}(A) = \mathcal{R}(A^H) \)
\( \mathcal{R}(A^H A) = \mathcal{R}(A) \) where \( M^\perp \equiv \{x \mid <x, m> = 0 \ \forall m \in M \} \)

c) Argue that the Normal Equations always have a solution \( x_0 \).
Will the solution be unique? If not, give an example.

d) Assume rank \( A = n \). Show that \( P = A (A^H A)^{-1} A^H \).

e) Show that \( P^H = P \) so that \( P^2 = P \).

f) Show that Least Square Error (LSE) is given by:
\[
E = \| b - A x_0 \| = \sqrt{b^H (I - P) b}
\]

g) Find the Least Square Error (LSE) line \( y = m_0 x + b_0 \) which "best" fits through the points \((x_1, y_1), \ldots, (x_k, y_k)\). Show that \( m_0 \) and \( b_0 \) satisfy:
\[
\begin{bmatrix}
  k & \sum x_i \\
  \sum x_i & \sum x_i^2
\end{bmatrix}
\begin{bmatrix}
  m_0 \\
  b_0
\end{bmatrix}
= \begin{bmatrix}
  \sum y_i \\
  \sum x_i y_i
\end{bmatrix}
\]
Consider the system
\[ x(t) = A x(t) + B \xi(t) \quad x \in \mathbb{C}^n \]
\[ y(t) = C x(t) + D \xi(t) \quad y \in \mathbb{C}^p \]

Define
\[ \Theta = \begin{bmatrix} C \\ CA \\ \vdots \\ C A^{n-1} \end{bmatrix} \quad \Phi = \begin{bmatrix} y(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} \]

\[ T = \begin{bmatrix} D \\ CB \\ \vdots \\ CAB \\ \vdots \\ C A^{n-2} B \end{bmatrix} \quad \Xi = \begin{bmatrix} \xi(t) \\ \vdots \\ \xi^{(n-1)}(t) \end{bmatrix} \]

1. Show that \( y(t) = \Theta x(t) + T \Xi(t) \)

2. Show that
Given \( y(t) \) and \( \Phi(t) \), we can determine \( x(t) \) uniquely if
\( \text{rank } \Theta = n \) (full column rank)

In such a case
\[ x(t) = (\Theta^T \Theta)^{-1} \Theta^T \{ y(t) - T \Xi(t) \} \]

Why?

3. What are some practical problems with obtaining \( x(t) \) in the above manner? (Hint: Think of \( \xi \) as noise)