

Problem 1 (Dynamic Augmentation: State Space Representation)

Consider a dynamical system consisting of several subsystems. The describing equations are as follows:

$$\begin{aligned} y_1 &= x_{a_1}, \dot{x}_{a_1} = y_{p_1} \\ y_2 &= y_{p_2} \\ y_p &= C_p x_p + D_p u_p, \dot{x}_p = A_p x_p + B_p u_p \\ u_{p_1} &= u_1, u_{p_2} = x_{a_2}, \dot{x}_{a_2} = -x_{a_2} + u_2 \end{aligned}$$

Sketch a block diagram and determine a state space representation for the above dynamical system. ■

Problem 2 (Gaussian Elimination, Fundamental Spaces, Least Squares, Minimum Norm)

Consider the following linear algebraic system of equations:

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad (1)$$

- (a) Determine the general solution.
 (b) Determine a basis for the four fundamental subspaces: $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$.

Now let $b = [1 \ -1 \ 1]^T$ and consider the vector norm $\|z\| \stackrel{\text{def}}{=} \sqrt{z^T z}$.

- (c) Determine the set of all possible x that minimizes the error $\|b - Ax\|_2$.
 (d) Determine the projection $P_{R(A)} b$ of the vector b onto the range of A .
 (e) Determine the minimum norm x which minimizes the error $\|b - Ax\|_2$. ■

Problem 3 (Controllability, State Transfer, Pole Placement)

Consider the LTI system defined by the state space dual $A = \begin{bmatrix} -p_1 & z - p_2 \\ 0 & -p_2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $z \neq p_1$ and $z \neq p_2$. Sketch a block diagram for the system.

- (a) Is the system controllable? Explain your answer. Are there any pole-zero cancellations?
 (b) Does there exist a control law which will transfer the state of the system from $x_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

If so, show how to determine one. Also, determine a minimum energy state transferring control law.

- (c) Does there exist a full state feedback control law $u = -Gx$ such that the closed loop system has poles at $s = -1 \pm j1$? If so, determine a suitable control gain matrix G . If not, explain why. ■

Problem 4 (Controllability, State Transfer, Pole Placement)

Consider the LTI system defined by the state space dual $A = \begin{bmatrix} -p_1 & z - p_2 \\ 0 & -p_2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $z = p_1$ and $z \neq p_2$.

- (a) Is the system controllable? Explain your answer. Are there any pole-zero cancellations?
 (b) Assuming $x_o = 0$, determine an expression for $x(t)$ using modal analysis concepts. Determine a basis for the set of states that are reachable from $x_o = 0$.

- (c) Does there exist a control law which will transfer the state of the system from $x_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?
 If so, determine one. Also, determine a minimum energy state transferring control law.

- (d) Does there exist a control law which will transfer the state of the system from $x_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

If so, determine one. If not, determine what state closest to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is reachable. Then determine a control law which will achieve the transfer to this reachable state. Is your control law unique? If not, show how to determine another state transferring control law and determine the minimum energy state transferring control law.

- (e) Does there exist a full state feedback control law $u = -Gx$ such that the closed loop system has poles at $s = -p_1, -2$ (where $p_1 \neq 2$)? If so, determine a suitable control gain matrix G . If not, explain why.

- (f) Does there exist a full state feedback control law $u = -Gx$ such that the closed loop system has poles at

$s = -2, -5$ (both distinct from $-p_1$)? If so, determine a suitable control gain matrix G . If not, explain why. ■

Problem 5 (Observability, State Reconstruction, Pole Placement)

Consider the LTI system defined by the state space triple $A = \begin{bmatrix} -p_1 & 1 \\ 0 & -p_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [z - p_1 \quad 1]$ where $z \neq p_1$ and $z \neq p_2$. Sketch a block diagram for the system.

- (a) Is the system observable? Explain your answer. Are there any pole-zero cancellations?
- (b) Suppose that $u = 0$ and y is known on $t \in [0, 1]$. Determine an expression for the set of all possible initial conditions x_o . Can x_o be determined uniquely? Explain. Determine the minimum norm initial condition.
- (c) Can one design an observer with closed loop poles at $s = -1, -2$? If not, explain why. If so, determine a suitable observer gain matrix H . Moreover, determine the state estimation error $\tilde{x} \stackrel{\text{def}}{=} x - \hat{x}$ when $u = 0$, the initial system state is $x_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the initial state estimate (used in your observer) is $\hat{x}_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. ■

Problem 6 (Observability, State Reconstruction, Pole Placement)

Consider the LTI system defined by the state space triple $A = \begin{bmatrix} -p_1 & 1 \\ 0 & -p_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [z - p_1 \quad 1]$ where $z \neq p_1$ and $z = p_2$.

- (a) Is the system observable? Explain your answer. Are there any pole-zero cancellations?
- (b) Suppose that $u = 0$ and y is known on $t \in [0, 1]$. Determine an expression for the set of all possible initial conditions x_o . Can x_o be determined uniquely? Explain. Determine the minimum norm initial condition.
- (c) Can one design an observer with closed loop poles at $s = -p_2, -2$ (where $p_2 \neq 2$)? If so, determine a suitable observer gain matrix H . If not, explain why.
- (d) Can one design an observer with closed loop poles at $s = -2, -5$ (both distinct from $-p_2$)? If so, determine a suitable observer gain matrix H . If not, explain why.
- (e) Discuss how the answers to (a)-(d) change if $z = p_1$ and $z \neq p_2$? Hint: Examine block diagram. ■

Problem 7 (Model Based Compensator Design)

Consider the linear time invariant plant

$$P(s) = \frac{2}{s-5}$$

with state space quadruple $A_p = 5$, $B_p = 1$, $C_p = 2$, $D_p = 0$.

- (a) Show how to design a model based compensator which satisfies the following design specifications:
 - (i) zero steady state error to step reference commands,
 - (ii) closed loop poles at $s = -4 \pm j3$, $s = -100 \pm j100$.
- (b) Discuss how one might minimize the overshoot due to a step reference command.
- (c) Summarize the design process if one desires to follow sinusoidal reference commands with frequency ω_o . ■

Problem 1 (Dynamic Augmentation: State Space Representation)

Consider a dynamical system consisting of several subsystems. The describing equations are as follows:

$$y_1 = x_{a_1}, \dot{x}_{a_1} = y_{p_1}$$

$$y_2 = y_{p_2}$$

$$y_p = C_p x_p + D_p u_p, \dot{x}_p = A_p x_p + B_p u_p$$

$$u_{p_1} = u_1, u_{p_2} = x_{a_2}, \dot{x}_{a_2} = -x_{a_2} + u_2$$

Sketch a block diagram and determine a state space representation for the above dynamical system. ■

Problem 1 - Partial Solution:

$$u_p = [u_1 \ x_{a_2}]^T \quad (2)$$

$$\dot{x}_{a_1} = y_{p_1} = [1 \ 0] (C_p x_p + D_p u_p) \quad (3)$$

$$\dot{x}_p = A_p x_p + B_p u_p \quad (4)$$

$$\dot{x}_{a_2} = -x_{a_2} + u_2 \quad (5)$$

$$y_1 = x_{a_1} \quad (6)$$

$$y_2 = y_{p_2} = [0 \ 1] (C_p x_p + D_p u_p) \quad (7)$$

$$B_p = [B_{p_1} \ B_{p_2}] \quad (8)$$

$$D_p = [D_{p_1} \ D_{p_2}] \quad (9)$$

$$\begin{bmatrix} \dot{x}_{a_1} \\ \dot{x}_p \\ \dot{x}_{a_2} \end{bmatrix} = \begin{bmatrix} 0 & [1 \ 0]C_p & [1 \ 0]B_{p_2} \\ 0 & A_p & B_{p_2} \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{a_1} \\ x_p \\ x_{a_2} \end{bmatrix} + \begin{bmatrix} [1 \ 0]B_{p_1} & 0 \\ B_{p_1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (10)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & [0 \ 1]C_p & [0 \ 1]D_{p_2} \end{bmatrix} \begin{bmatrix} x_{a_1} \\ x_p \\ x_{a_2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [0 \ 1]D_{p_1} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (11)$$

Problem 2 (Gaussian Elimination, Fundamental Spaces, Least Squares, Minimum Norm)

Consider the following linear algebraic system of equations:

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad (12)$$

(a) Determine the general solution.

(b) Determine a basis for the four fundamental subspaces: $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$.

(c) Determine the minimum norm solution.

Now let $b = [1 \ -1 \ 1]^T$ and consider the vector norm $\|z\| \stackrel{\text{def}}{=} \sqrt{z^T z}$.

(d) Determine the set of all possible x that minimizes the error $\|b - Ax\|_2$.

(e) Determine the projection $P_{R(A)} b$ of the vector b onto the range of A .

(f) Determine the minimum norm x which minimizes the error $\|b - Ax\|_2$. ■

Problem 2 Solution:

(a) Gaussian Elimination yields

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \hat{b} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - 2b_1 \end{bmatrix} \quad (13)$$

From this, it follows that $\text{rank}(A) = 2$. This is also the number of basic variables. The basic variables are x_1 and x_4 . From the last equation, we have $x_4 = b_3 - 2b_1$. From this and the the first equation, it follows that $x_1 = b_1 - 2x_4 = b_1 - 2(b_3 - 2b_1) = 5b_1 - 2b_3$. The general solution is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ 0 \\ 0 \\ -2b_1 + b_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 \quad (14)$$

$$[-1 \ 1 \ 0] b = 0 \quad \leftarrow \text{constraint on } b; \text{ required for existence} \quad (15)$$

Check, via substitution into Ax , that the above general solution does in fact satisfy $Ax = b = [b_1 \ b_2 \ b_3]^T$.

(b) Dimensions and bases for each of the 4 fundamnntal subspaces are as follows:

- From the above Gaussian elimination, it follows that $\dim R(A) = \text{rank}(A) = 2$. A basis for $R(A)$ is:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}. \quad (16)$$

- Similarly, $\dim R(A^H) = \text{rank}(A) = 2$. A basis for $R(A^H)$ is as follows:

$$[1 \ 0 \ 0 \ 2], [2 \ 0 \ 0 \ 5] \quad (17)$$

or

$$[1 \ 0 \ 0 \ 2], [0 \ 0 \ 0 \ 1]. \quad (18)$$

- From the above Gaussian elimination, it follows that $\dim N(A) = n - \text{rank}(A) = 4 - 2 = 2$ (number of free variables). A basis for $N(A)$ is given by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \quad (19)$$

- From the above Gaussian elimination, it follows that $\dim N(A^H) = m - \text{rank}(A) = 3 - 2 = 1$ (number of constraints on b). A basis for $N(A^H)$ is given by

$$[-1 \ 1 \ 0]. \quad (20)$$

This follows from the constraint equation (Equation (15)) that resulted from the above Gaussian elimination process.

(c) To determine the minimum norm solution (which is unique), we need to determine a solution that lies in the row space of A (or row space of U). Two approaches are used to address this component of the problem.

1. *Approach 1: Using U and \hat{b} .* It suffices to solve $UU^H z = \hat{b}$ for any z and choose $x = U^H z$. Doing so yields

$$UU^H z = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} z = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} z = \hat{b} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - 2b_1 \end{bmatrix}. \quad (21)$$

Performing Gaussian elimination on this yields

$$\begin{bmatrix} 10 & 0 & 4 \\ 0 & 0 & 0 \\ 10 & 0 & 5 \end{bmatrix} z = \begin{bmatrix} 2b_1 \\ b_2 - b_1 \\ 5b_3 - 10b_1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0.4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} z = \begin{bmatrix} 0.2b_1 \\ b_2 - b_1 \\ 5b_3 - 12b_1 \end{bmatrix}. \quad (22)$$

From this, it follows that z_2 is free, z_1 and z_3 are basic, $z_3 = -12b_1 + 5b_3$, $z_1 = 0.2b_1 - 0.4z_3 = 0.2b_1 - 0.4(-12b_1 + 5b_3) = 0.2b_1 + 4.8b_1 - 2b_3 = 5b_1 - 2b_3$. The general z solution is then

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ 0 \\ -12b_1 + 5b_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} z_2. \quad (23)$$

From this, we obtain the minimum norm solution

$$x = U^H z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 5b_1 - 2b_3 \\ 0 \\ -12b_1 + 5b_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} z_2 \right) \quad (24)$$

$$= \begin{bmatrix} 5b_1 - 2b_3 \\ 0 \\ 0 \\ 10b_1 - 4b_3 - 12b_1 + 5b_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ 0 \\ 0 \\ -2b_1 + b_3 \end{bmatrix} \quad (25)$$

2. *Approach 2: Using A and b .* Note that using the pair $AA^H w = b$, $x = A^H w$, yields the same result; but it generally involves more cumbersome arithmetic. Lets show this:

$$AA^H w = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 5 \end{bmatrix} w = \begin{bmatrix} 5 & 5 & 12 \\ 5 & 5 & 12 \\ 12 & 12 & 29 \end{bmatrix} w = b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (26)$$

Gaussian elimination on this yields

$$\begin{bmatrix} 60 & 60 & 144 \\ 0 & 0 & 0 \\ 60 & 60 & 145 \end{bmatrix} w = b = \begin{bmatrix} 12b_1 \\ b_2 - b_1 \\ 5b_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & \frac{144}{60} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} w = b = \begin{bmatrix} 0.2b_1 \\ b_2 - b_1 \\ 5b_3 - 12b_1 \end{bmatrix} \quad (27)$$

From this, we obtain $w_3 = 5b_3 - 12b_1$ and $w_1 = 0.2b_1 - \frac{144}{60}w_3 - w_2 = 0.2b_1 - \frac{144}{60}(5b_3 - 12b_1) - w_2 = 0.2b_1 - 12b_3 + 28.8b_1 - w_2 = 29b_1 - 12b_3 - w_2$. From this, it follows that

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 29b_1 - 12b_3 \\ 0 \\ -12b_1 + 5b_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} w_2. \quad (28)$$

This yields the minimum norm solution

$$x = A^H w = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 5 \end{bmatrix} \left(\begin{bmatrix} 29b_1 - 12b_3 \\ 0 \\ -12b_1 + 5b_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} w_2 \right) = \begin{bmatrix} 29b_1 - 12b_3 - 24b_1 + 10b_3 \\ 0 \\ 0 \\ 58b_1 - 24b_3 - 60b_1 + 25b_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ 0 \\ 0 \\ -2b_1 + b_3 \end{bmatrix} \quad (29)$$

It should be noted that the minimum norm solution is the unique projection of any solution onto the row space of A (or U).

(d) To determine the set of all vectors x that minimize $\|b - Ax\|$, it is necessary and sufficient to determine the set of all vectors x that satisfy the normal equations $A^H Ax = A^H b$. To do so, we proceed as follows:

$$A^H Ax = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 5 \end{bmatrix} x = \begin{bmatrix} 6 & 1 & 2 & 14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 14 & 0 & 0 & 33 \end{bmatrix} x = A^H b = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (30)$$

This yields

$$A^H Ax = \begin{bmatrix} 6 & 1 & 2 & 14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 14 & 0 & 0 & 33 \end{bmatrix} x = A^H b = \begin{bmatrix} b_1 + b_2 + 2b_3 \\ 0 \\ 0 \\ 2b_1 + 2b_2 + 5b_3 \end{bmatrix} \quad (31)$$

From this, it follows that

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 & 14 \\ 14 & 33 \end{bmatrix}^{-1} \begin{bmatrix} b_1 + b_2 + 2b_3 \\ 2b_1 + 2b_2 + 5b_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 33 & -14 \\ -14 & 6 \end{bmatrix} \begin{bmatrix} b_1 + b_2 + 2b_3 \\ 2b_1 + 2b_2 + 5b_3 \end{bmatrix} \quad (32)$$

$$= \begin{bmatrix} 16.5 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} b_1 + b_2 + 2b_3 \\ 2b_1 + 2b_2 + 5b_3 \end{bmatrix} \quad (33)$$

$$= \begin{bmatrix} 16.5(b_1 + b_2 + 2b_3) - 7(2b_1 + 2b_2 + 5b_3) \\ -7(b_1 + b_2 + 2b_3) + 3(2b_1 + 2b_2 + 5b_3) \end{bmatrix} \quad (34)$$

$$= \begin{bmatrix} 2.5b_1 + 2.5b_2 - 2b_3 \\ -b_1 - b_2 + b_3 \end{bmatrix} \quad (35)$$

The general solution is then given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.5b_1 + 2.5b_2 - 2b_3 \\ 0 \\ 0 \\ -b_1 - b_2 + b_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 \quad (36)$$

(e) To determine the unique projection of b onto $R(A)$, we need any solution to the normal equations $A^H Ax = A^H b$ and then $P_{R(A)} b = Ax$. Using the result from (d) yields

$$P_{R(A)} b = Ax = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 5 \end{bmatrix} \left(\begin{bmatrix} 2.5b_1 + 2.5b_2 - 2b_3 \\ 0 \\ 0 \\ -b_1 - b_2 + b_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 \right) \quad (37)$$

$$= \begin{bmatrix} 1(2.5b_1 + 2.5b_2 - 2b_3) + 2(-b_1 - b_2 + b_3) \\ 1(2.5b_1 + 2.5b_2 - 2b_3) + 2(-b_1 - b_2 + b_3) \\ 2(2.5b_1 + 2.5b_2 - 2b_3) + 5(-b_1 - b_2 + b_3) \end{bmatrix} = \begin{bmatrix} 0.5b_1 + 0.5b_2 \\ 0.5b_1 + 0.5b_2 \\ b_3 \end{bmatrix} \quad (38)$$

Recall that $b \in R(A)$ if and only if $b_2 = b_1$. In such a case, we note that $P_{R(A)}b = [b_1 \ b_2 \ b_3]$ as expected.

(f) To determine the minimum norm x that minimizes $\|b - Ax\|$, we must solve $AA^H z = P_{R(A)}b$ for any z and then take $x = A^H z$. Doing so yields

$$AA^H z = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 5 \end{bmatrix} z = \begin{bmatrix} 5 & 5 & 12 \\ 5 & 5 & 12 \\ 12 & 12 & 29 \end{bmatrix} w = P_{R(A)}b = \begin{bmatrix} 0.5b_1 + 0.5b_2 \\ 0.5b_1 + 0.5b_2 \\ b_3 \end{bmatrix} \quad (39)$$

Gaussian elimination yields

$$\begin{bmatrix} 60 & 60 & 144 \\ 0 & 0 & 0 \\ 60 & 60 & 145 \end{bmatrix} z = \begin{bmatrix} 6b_1 + 6b_2 \\ 0 \\ 5b_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & \frac{144}{60} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} z = P_{R(A)}b = \begin{bmatrix} 0.1b_1 + 0.1b_2 \\ 0 \\ -6b_1 - 6b_2 + 5b_3 \end{bmatrix} \quad (40)$$

The latter equations yield $z_3 = -6b_1 - 6b_2 + 5b_3$ and $z_1 = 0.1b_1 + 0.1b_2 - \frac{144}{60}z_3 - z_2 = 0.1b_1 + 0.1b_2 + 14.4b_1 + 14.4b_2 - 12b_3 - z_2 = 14.5b_1 + 14.5b_2 - 12b_3 - z_2$. From this, we obtain the general solution

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 14.5b_1 + 14.5b_2 - 12b_3 \\ 0 \\ -6b_1 - 6b_2 + 5b_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} z_2 \quad (41)$$

From this, the minimum norm x that minimizes $\|b - Ax\|$ is given by

$$x = A^H z = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 5 \end{bmatrix} \left(\begin{bmatrix} 14.5b_1 + 14.5b_2 - 12b_3 \\ 0 \\ -0.5b_1 - 0.5b_2 + 5b_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} z_2 \right) \quad (42)$$

$$= \begin{bmatrix} 1(14.5b_1 + 14.5b_2 - 12b_3) + 2(-6b_1 - 6b_2 + 5b_3) \\ 0 \\ 0 \\ 2(14.5b_1 + 14.5b_2 - 12b_3) + 5(-6b_1 - 6b_2 + 5b_3) \end{bmatrix} = \begin{bmatrix} 2.5b_1 + 2.5b_2 - 2b_3 \\ 0 \\ 0 \\ -b_1 - b_2 + b_3 \end{bmatrix} \quad (43)$$

Finally, it is worth noting that if $b_2 = b_1$ then $b \in R(A)$, $P_{R(A)}b = b$, and the above minimum norm x that minimizes $Ax = b$ takes the form $x = A^H z = [5b_1 - 2b_3 \ 0 \ 0 \ -2b_1 + b_3]^T$ which is in agreement with the answer obtained in (c). ■

Problem 3 (Controllability, State Transfer, Pole Placement)

Consider the LTI system defined by the state space dual $A = \begin{bmatrix} -p_1 & z - p_2 \\ 0 & -p_2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $z \neq p_1$ and $z \neq p_2$. Sketch a block diagram for the system.

(a) Is the system controllable? Explain your answer. Are there any pole-zero cancellations?

(b) Does there exist a control law which will transfer the state of the system from $x_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

If so, show how to determine one. Also, determine a minimum energy state transferring control law.

(c) Does there exist a full state feedback control law $u = -Gx$ such that the closed loop system has poles at $s = -1 \pm j1$? If so, determine a suitable control gain matrix G . If not, explain why. ■

Problem 3 - Partial Solution:

(a) From the block diagram, one can identify two systems in series. One system (the input system) has

transfer function $\frac{z-p_2}{s+p_2} + 1 = \frac{s+z}{s+p_2}$. This system feeds into a system (the output system) with transfer function $\frac{1}{s+p_1}$. The transfer function of the composite system is therefore $\frac{1}{s+p_1} \left[\frac{s+z}{s+p_2} \right]$. From this, one sees that we have a pole-zero cancellation, the state x_1 is uncontrollable, and $-p_1$ is an uncontrollable mode if and only if $z = p_1$. The controllability matrix is given by

$$\mathcal{C} = [B \ AB] = \begin{bmatrix} 1 & z - p_1 - p_2 \\ 1 & -p_2 \end{bmatrix} \quad (44)$$

From this, we see that $\det \mathcal{C} = -p_2 + p_2 + p_1 - z = p_1 - z$. Given this, the system is uncontrollable if and only if $z \neq p_1$. Since $z \neq p_1$, it follows that the system is controllable and there are no pole-zero cancellations. It should be noted that the condition $z \neq p_2$ is NOT needed for controllability.

(b) Since the system is controllable, there exists a control u that will transfer the state from any initial state $x(t_o) = x_o$ to any final state $x(t_f) = x_f$ in finite time. Such a state transferring control law is provided by any u satisfying

$$\left[e^{A(t_o-t_f)} x_f - x_o \right] = \int_{t_o}^{t_f} e^{A(t_o-\tau)} B u(\tau) d\tau. \quad (45)$$

Such a u is given by

$$u = M(t) \left(\int_{t_o}^{t_f} e^{A(t_o-\tau)} B M(\tau) d\tau \right)^{-1} \left[e^{A(t_o-t_f)} x_f - x_o \right] \quad (46)$$

where $M(t)$ is any matrix-valued function of t such that the above inverse exists. A particular control law that does the job is the minimum energy control law. This control law is obtained when $M(t) = B^H e^{A^H(t_o-t)}$. In such a case, we have the following Gramian formula:

$$u = B^H e^{A^H(t_o-t)} \mathcal{G}_c(t_o, t_f)^{-1} \left[e^{A(t_o-t_f)} x_f - x_o \right] \quad (47)$$

where

$$\mathcal{G}_c(t_o, t_f) = \int_{t_o}^{t_f} e^{A(t_o-\tau)} B B^H e^{A^H(t_o-\tau)} d\tau. \quad (48)$$

For our problem, e^{At} may be computed as follows

$$(sI - A)^{-1} = \begin{pmatrix} s + p_1 & p_2 - z \\ 0 & s + p_2 \end{pmatrix}^{-1} = \frac{1}{(s+p_1)(s+p_2)} \begin{pmatrix} s + p_2 & z - p_2 \\ 0 & s + p_1 \end{pmatrix} \quad (49)$$

$$= \begin{pmatrix} \frac{1}{s+p_1} & \frac{z-p_2}{(s+p_1)(s+p_2)} \\ 0 & \frac{1}{s+p_2} \end{pmatrix}. \quad (50)$$

From this, it follows that

$$e^{At} = \begin{pmatrix} e^{-p_1 t} & \frac{z-p_2}{p_2-p_1} [e^{-p_1 t} - e^{-p_2 t}] \\ 0 & e^{-p_2 t} \end{pmatrix}. \quad (51)$$

(c) Since the system is controllable, there exists a full state feedback control gain matrix G which will place the eigenvalues of $A - BG$ anywhere in the complex plane - modulo complex conjugate constraints. For our problem,

$$A - BG = \begin{bmatrix} -p_1 & z - p_2 \\ 0 & -p_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [g_1 \ g_2] = \begin{bmatrix} -p_1 - g_1 & z - p_2 - g_2 \\ -g_1 & -p_2 - g_2 \end{bmatrix}. \quad (52)$$

To place closed loop poles at $-1 \pm j1$, we require

$$\det(sI - A + BG) = \det \begin{bmatrix} s + p_1 + g_1 & g_2 + p_2 - z \\ g_1 & s + p_2 + g_2 \end{bmatrix} \quad (53)$$

$$= s^2 + (p_1 + g_1 + p_2 + g_2)s + (p_1 + g_1)(p_2 + g_2) + g_1(z - p_2 - g_2) \quad (54)$$

$$= s^2 + (g_1 + g_2 + p_1 + p_2)s + p_1 p_2 + g_1(p_2 + z - p_2) + g_2 p_1 \quad (55)$$

$$= s^2 + 2s + 2. \quad (56)$$

The above yields the algebraic system of equations:

$$\begin{bmatrix} 1 & 1 \\ z & p_1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 2 - p_1 - p_2 \\ 2 - p_1 p_2 \end{bmatrix}. \quad (57)$$

As expected, this system possesses a solution if and only if $z \neq p_1$; i.e. if and only if the system is controllable. In such a case, the solution is given by

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ z & p_1 \end{bmatrix}^{-1} \begin{bmatrix} 2 - p_1 - p_2 \\ 2 - p_1 p_2 \end{bmatrix} = \frac{1}{p_1 - z} \begin{bmatrix} p_1 & -1 \\ -z & 1 \end{bmatrix} \begin{bmatrix} 2 - p_1 - p_2 \\ 2 - p_1 p_2 \end{bmatrix} \quad (58)$$

$$= \frac{1}{p_1 - z} \begin{bmatrix} p_1(2 - p_1 - p_2) - 1(2 - p_1 p_2) \\ -z(2 - p_1 - p_2) + 1(2 - p_1 p_2) \end{bmatrix} \quad (59)$$

■

Problem 4 (Controllability, State Transfer, Pole Placement)

Consider the LTI system defined by the state space dual $A = \begin{bmatrix} -p_1 & z - p_2 \\ 0 & -p_2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $z = p_1$ and $z \neq p_2$.

(a) Is the system controllable? Explain your answer. Are there any pole-zero cancellations?

(b) Assuming $x_o = 0$, determine an expression for $x(t)$ using modal analysis concepts. Determine a basis for the set of states that are reachable from $x_o = 0$.

(c) Does there exist a control law which will transfer the state of the system from $x_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

If so, determine one. Also, determine a minimum energy state transferring control law.

(d) Does there exist a control law which will transfer the state of the system from $x_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

If so, determine one. If not, determine what state closest to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is reachable. Then determine a control law which will achieve the transfer to this reachable state. Is your control law unique? If not, show how to determine another state transferring control law and determine the minimum energy state transferring control law.

(e) Does there exist a full state feedback control law $u = -Gx$ such that the closed loop system has poles at $s = -p_1, -2$ (where $p_1 \neq 2$)? If so, determine a suitable control gain matrix G . If not, explain why.

(f) Does there exist a full state feedback control law $u = -Gx$ such that the closed loop system has poles at $s = -2, -5$ (both distinct from $-p_1$)? If so, determine a suitable control gain matrix G . If not, explain why. ■

Problem 4 - Solution:

(a) From the block diagram, one sees that we have a pole-zero cancellation, the state x_1 is uncontrollable, and $-p_1$ is an uncontrollable mode if and only if $z = p_1$. The controllability matrix is given by

$$\mathcal{C} = [B \ AB] = \begin{bmatrix} 1 & z - p_1 - p_2 \\ 1 & -p_2 \end{bmatrix} \quad (60)$$

From this, we see that $\det \mathcal{C} = -p_2 + p_2 + p_1 - z = p_1 - z$. Given this, the system is uncontrollable if and only if $z = p_1$. Since $z = p_1$, it follows that the system is uncontrollable and there is a pole-zero cancellation at $s = p_1$. It should be noted that the condition $z \neq p_2$ is NOT needed for controllability.

We now use PBH eigenvalue-eigenvector ideas to show that p_1 is an uncontrollable mode. Since A is diagonal, it follows that the system eigenvalues can be determined by inspection to be $\lambda_1 = -p_1$ and $\lambda_2 = -p_2$. We

note that that $(sI - A) = \begin{bmatrix} s + p_1 & p_2 - p_1 \\ 0 & s + p_2 \end{bmatrix}$.

- For $s = -p_1$, we have $(sI - A) = \begin{bmatrix} 0 & p_2 - p_1 \\ 0 & p_2 - p_1 \end{bmatrix}$. It follows that $v_1 = [1 \ 0]^T$ is an eigenvector associated with $-p_1$.

- For $s = -p_2$, we have $(sI - A) = \begin{bmatrix} p_1 - p_2 & p_2 - p_1 \\ 0 & 0 \end{bmatrix}$. It follows that $v_2 = [1 \ 1]^T$ is an eigenvector associated with $-p_2$.

The matrix of right eigenvectors is given by

$$V = [v_1 \ v_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (61)$$

Since the eigenvectors are linearly independent, this matrix is invertible, and the system A matrix is said to be diagonalizable. The corresponding matrix of left eigenvectors is then given by

$$W = \begin{bmatrix} -w_1^H & - \\ -w_2^H & - \end{bmatrix} = V^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \quad (62)$$

Next, we note that $w_1^H B = [1 \ -1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$. It thus follows from the PBH eigenvalue-eigenvector modal controllability test that $-p_1$ is an uncontrollable mode. This, of course, was expected from the pole-zero cancellation within the system block diagram.

(b) To determine an expression for $x(t)$, we use the above modal information. Specifically, we note that

$$e^{At} B = e^{\lambda_1 t} v_1 w_1^H B + e^{\lambda_2 t} v_2 w_2^H B = e^{\lambda_2 t} v_2 w_2^H B = e^{-p_2 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [0 \ 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-p_2 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (63)$$

It thus follows that

$$x(1) = x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \int_0^1 e^{-p_2(1-\tau)} u(\tau) d\tau. \quad (64)$$

A basis for the set of all reachable states from $x_o = 0$ is then $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) There exists a control law which transfers the state from $x_o = 0$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In fact, any control law satisfying

$$1 = \int_0^1 e^{-p_2(1-\tau)} u(\tau) d\tau = e^{-p_2} \int_0^1 e^{p_2\tau} u(\tau) d\tau \quad (65)$$

will do the job. The following is such a control law:

$$u = \left[\frac{1}{\int_0^1 e^{p_2\tau} d\tau} \right] e^{p_2} \quad (66)$$

where it is assumed that the integral in the denominator is non-zero. This will be the case for any finite p_2 . One can also determine the minimum energy control law that will perform the desired state transfer. To do so, we apply our minimum energy control law Gramian formula with $x_o = 0$, $t_o = 0$, $x_f = 1$, $t_f = 1$, $A = -p_2$, $B = 1$. This works because $x_f = \int_{t_o}^{t_f} e^{A(t_f-\tau)} B u(\tau) d\tau$ matches the above (desired) integral constraint for the selected values. For the above selected values, we get the following from our minimum energy Gramian formula

$$\mathcal{G}_c(t_o, t_f) = \int_{t_o}^{t_f} e^{A(t_o-\tau)} B B^H e^{A^H(t_o-\tau)} d\tau = \int_0^1 e^{2p_2\tau} d\tau = \frac{1}{2p_2} [e^{2p_2} - 1] \quad (67)$$

$$u = B^H e^{A^H(t_o-t)} \mathcal{G}_c(t_o, t_f)^{-1} [e^{A(t_o-t_f)} x_f - x_o] = e^{p_2 t} \frac{p_2}{e^{2p_2} - 1} [e^{p_2}] \quad (68)$$

for $p_2 \neq 0$. For $p_2 = 0$, we obtain

$$\mathcal{G}_c(t_o, t_f) = \int_{t_o}^{t_f} e^{A(t_o-\tau)} B B^H e^{A^H(t_o-\tau)} d\tau = \int_0^1 e^{2p_2\tau} d\tau = 1 \quad (69)$$

$$u = B^H e^{A^H(t_o-t)} \mathcal{G}_c(t_o, t_f)^{-1} [e^{A(t_o-t_f)} x_f - x_o] = e^{p_2 t} [e^{p_2}]. \quad (70)$$

(d) Given the analysis in (b), it follows that there does not exist a control law that will transfer the state from $x_o = 0$ to $x(1) = x_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not lie along the span of $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In such a case, we can determine a control law that will minimize $\|b - Ax\| = \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} x \right\|$ where $x = \int_0^1 e^{-p_2(1-\tau)} u(\tau) d\tau$. To determine x , we solve the normal equations $A^H Ax = A^H b$ or $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This yields $2x = 1$ or $x = \frac{1}{2}$. We thus need to determine a u such that $\frac{1}{2} = \int_0^1 e^{-p_2(1-\tau)} u(\tau) d\tau$. One u that does the job is $u = \left[\frac{1}{\int_0^1 e^{p_2\tau} d\tau} \right] e^{p_2 \frac{1}{2}}$. There are infinitely many control laws that will do the job. Using $x_o = 0$, $t_o = 0$, $x_f = \frac{1}{2}$, $t_f = 1$, $A = -p_2$, $B = 1$, the minimum energy control law is given by

$$\mathcal{G}_c(t_o, t_f) = \int_{t_o}^{t_f} e^{A(t_o-\tau)} B B^H e^{A^H(t_o-\tau)} d\tau = \int_0^1 e^{2p_2\tau} d\tau = \frac{1}{2p_2} [e^{2p_2} - 1] \quad (71)$$

$$u = B^H e^{A^H(t_o-t)} \mathcal{G}_c(t_o, t_f)^{-1} \left[e^{A(t_o-t_f)} x_f - x_o \right] = e^{p_2 t} \frac{p_2}{e^{2p_2} - 1} \left[e^{p_2} \frac{1}{2} \right] \quad (72)$$

for $p_2 \neq 0$. For $p_2 = 0$, we obtain

$$\mathcal{G}_c(t_o, t_f) = \int_{t_o}^{t_f} e^{A(t_o-\tau)} B B^H e^{A^H(t_o-\tau)} d\tau = \int_0^1 e^{2p_2\tau} d\tau = 1 \quad (73)$$

$$u = B^H e^{A^H(t_o-t)} \mathcal{G}_c(t_o, t_f)^{-1} \left[e^{A(t_o-t_f)} x_f - x_o \right] = e^{p_2 t} \left[e^{p_2} \frac{1}{2} \right]. \quad (74)$$

(e) Since $-p_1$ is an uncontrollable mode, it cannot be moved via full state feedback (linear or otherwise). Given this, it is possible to find a control gain matrix G such that $A - BG$ has eigenvalues at $-p_1$ and -2 where $p_1 \neq 2$. That is, one pole stays fixed at $-p_1$; the other can be moved to any desired (real) location. With $z = p_1$, it follows that

$$\det(sI - A + BG) = \det \begin{bmatrix} s + p_1 + g_1 & g_2 + p_2 - p_1 \\ g_1 & s + p_2 + g_2 \end{bmatrix} \quad (75)$$

$$= s^2 + (p_1 + g_1 + p_2 + g_2)s + (p_1 + g_1)(p_2 + g_2) + g_1(p_1 - p_2 - g_2) \quad (76)$$

$$= s^2 + (g_1 + g_2 + p_1 + p_2)s + p_1(g_1 + g_2 + p_2) \quad (77)$$

$$= (s + p_1)(s + g_1 + g_2 + p_2). \quad (78)$$

This corroborates the above; namely that $-p_1$ cannot be moved; the other closed loop eigenvalue (pole) can be placed at any desired real location. $g_1 + g_2 + p_2 = 2$ will, for example, place the other pole at -2 .

(f) Since $-p_2$ is uncontrollable and hence cannot be moved, it is NOT possible to move both poles to arbitrary locations. If -2 and -5 are distinct from $-p_1$, then no G exists. If one of these are $-p_2$, then from above, a G can be found.

Problem 5 (Observability, State Reconstruction, Pole Placement)

Consider the LTI system defined by the state space triple $A = \begin{bmatrix} -p_1 & 1 \\ 0 & -p_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [z - p_1 \quad 1]$ where $z \neq p_1$ and $z \neq p_2$. Sketch a block diagram for the system.

(a) Is the system observable? Explain your answer. Are there any pole-zero cancellations?

(b) Suppose that $u = 0$ and y is known on $t \in [0, 1]$. Determine an expression for the set of all possible initial conditions x_o . Can x_o be determined uniquely? Explain. Determine the minimum norm initial condition.

(c) Can one design an observer with closed loop poles at $s = -1, -2$? If not, explain why. If so, determine a suitable observer gain matrix H . Moreover, determine the state estimation error $\tilde{x} \stackrel{\text{def}}{=} x - \hat{x}$ when $u = 0$, the initial system state is $x_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the initial state estimate (used in your observer) is $\hat{x}_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. ■

Problem 5 - Partial Solution:

(a) From the block diagram, one can identify two systems in series. One system (the input system) has transfer function $\frac{1}{s+p_2}$. This system feeds into a system (the output system) with transfer function $\frac{z-p_1}{s+p_1} + 1 = \frac{s+z}{s+p_1}$. The transfer function of the composite system is therefore $\left[\frac{s+z}{s+p_1}\right] \frac{1}{s+p_2}$. From this, one sees that we have a pole-zero cancellation, the state x_2 is unobservable, and $-p_2$ is an unobservable mode if and only if $z = p_2$.

Problem 6 (Observability, State Reconstruction, Pole Placement)

Consider the LTI system defined by the state space triple $A = \begin{bmatrix} -p_1 & 1 \\ 0 & -p_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [z - p_1 \quad 1]$

where $z \neq p_1$ and $z = p_2$.

- Is the system observable? Explain your answer. Are there any pole-zero cancellations?
- Suppose that $u = 0$ and y is known on $t \in [0, 1]$. Determine an expression for the set of all possible initial conditions x_o . Can x_o be determined uniquely? Explain. Determine the minimum norm initial condition.
- Can one design an observer with closed loop poles at $s = -p_2, -2$ (where $p_2 \neq 2$)? If so, determine a suitable observer gain matrix H . If not, explain why.
- Can one design an observer with closed loop poles at $s = -2, -5$ (both distinct from $-p_2$)? If so, determine a suitable observer gain matrix H . If not, explain why.
- Discuss how the answers to (a)-(d) change if $z = p_1$ and $z \neq p_2$? Hint: Examine block diagram. ■

Problem 7 (Model Based Compensator Design)

Consider the linear time invariant plant

$$P(s) = \frac{2}{s-5}$$

with state space quadruple $A_p = 5$, $B_p = 1$, $C_p = 2$, $D_p = 0$.

- Show how to design a model based compensator which satisfies the following design specifications:
 - zero steady state error to step reference commands,
 - closed loop poles at $s = -4 \pm j3$, $s = -100 \pm j100$.
- Discuss how one might minimize the overshoot due to a step reference command.
- Summarize the design process if one desires to follow sinusoidal reference commands with frequency ω_o . ■